

# A Size Upper Bound for Dominating Cycles

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## Abstract

Recently it was shown (by the author) that every graph of size  $q$  (the number of edges) and minimum degree  $\delta$  is hamiltonian if  $q \leq \delta^2 + \delta - 1$  (arXiv:1107.2201v1). In this paper we present the exact analog of this result for dominating cycles: if  $G$  is a 2-connected graph with  $q \leq 8$  if  $\delta = 2$  and  $q \leq (3(\delta - 1)(\delta + 2) - 1)/2$  if  $\delta \geq 3$ , then each longest cycle in  $G$  is a dominating cycle. The result is sharp in all respects.

Key words: Dominating cycle, size, minimum degree.

## 1 Introduction

Only finite undirected graphs without loops or multiple edges are considered. We reserve  $n$ ,  $q$ ,  $\delta$  and  $\kappa$  to denote the number of vertices (order), the number of edges (size), the minimum degree and the connectivity of a graph, respectively. A good reference for any undefined terms is [1].

The earliest sufficient condition for a graph to be hamiltonian was developed in 1952 due to Dirac [2] and is based on the natural idea that if a sufficient number of edges are present in the graph on  $n$  vertices (by keeping the minimum degree at a fairly high level) then a Hamilton cycle will exist.

**Theorem A [2].** Every graph with  $\delta \geq \frac{1}{2}n$  is hamiltonian.

A direct link between the number of edges and Hamilton cycles was established in 1959 due to Erdős and Gallai [3].

**Theorem B [3].** Every graph with  $q \geq \frac{1}{2}(n^2 - 3n + 5)$  is hamiltonian.

Recently it was proved a little surprising and, in fact, a contrary statement ensuring the existence of a Hamilton cycle if the number of edges is less than  $\delta^2 + \delta$ .

**Theorem C [5].** Every graph with  $q \leq \delta^2 + \delta - 1$  is hamiltonian.

In this paper we present the exact analog of Theorem C for dominating cycles.

**Theorem 1.** Let  $G$  be a 2-connected graph. If

$$q \leq \begin{cases} 8 & \text{if } \delta = 2, \\ \frac{3(\delta-1)(\delta+2)-1}{2} & \text{if } \delta \geq 3, \end{cases}$$

then each longest cycle in  $G$  is a dominating cycle.

To show that Theorem 1 is sharp, suppose first that  $\delta = 2$ . The graph  $K_1 + 2K_2$  shows that the connectivity condition  $\kappa \geq 2$  in Theorem 1 can not be relaxed by replacing it with  $\kappa \geq 1$ . The graph with vertex set  $\{v_1, v_2, \dots, v_8\}$  and edge set

$$\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_7, v_7v_8, v_8v_4\},$$

shows that the size bound  $q \leq 8$  can not be relaxed by replacing it with  $q \leq 9$ . Finally, the graph  $K_2 + 3K_1$  shows that the conclusion "each longest cycle in  $G$  is a dominating cycle" can not be strengthened by replacing it with " $G$  is hamiltonian". Now let  $\delta \geq 3$ . The graph  $K_1 + 2K_\delta$  shows that the connectivity condition  $\kappa \geq 2$  in Theorem 1 can not be relaxed by replacing it with  $\kappa \geq 1$ . Further, the graph  $K_2 + 3K_{\delta-1}$  shows that the size bound  $q \leq (3(\delta-1)(\delta+2)-1)/2$  can not be relaxed by replacing it with  $q \leq 3(\delta-1)(\delta+2)/2$ . Finally, the graph  $K_\delta + (\delta+1)K_1$  shows that the main conclusion "each longest cycle in  $G$  is a dominating cycle" can not be strengthened by replacing it with " $G$  is hamiltonian". So, Theorem 1 is best possible in all respects.

The following theorems are useful.

**Theorem D [2].** Every 2-connected graph either has a Hamilton cycle or has a cycle of length at least  $2\delta$ .

**Theorem E [4].** Let  $G$  be a graph,  $C$  a longest cycle in  $G$  and  $P$  a longest path in  $G \setminus C$  of length  $\bar{p}$ . Then  $|C| \geq (\bar{p} + 2)(\delta - \bar{p})$ .

**Theorem F [6].** Let  $G$  be a graph on  $n$  vertices and  $d(x) + d(y) \geq n$  for each nonadjacent vertices  $x, y$ . Then  $G$  is hamiltonian.

## 2 Notations and preliminaries

The set of vertices of a graph  $G$  is denoted by  $V(G)$  and the set of edges by  $E(G)$ . For  $S$  a subset of  $V(G)$ , we denote by  $G \setminus S$  the maximum subgraph of  $G$  with vertex set  $V(G) \setminus S$ . For a subgraph  $H$  of  $G$  we use  $G \setminus H$  short for  $G \setminus V(H)$ . The neighborhood of a vertex  $x \in V(G)$  will be denoted by  $N(x)$ . Set  $d(x) = |N(x)|$ . Furthermore, for a subgraph  $H$  of  $G$  and  $x \in V(G)$ , we define  $N_H(x) = N(x) \cap V(H)$  and  $d_H(x) = |N_H(x)|$ .

A simple cycle (or just a cycle)  $C$  of length  $t$  is a sequence  $v_1v_2\dots v_tv_1$  of distinct vertices  $v_1, \dots, v_t$  with  $v_iv_{i+1} \in E(G)$  for each  $i \in \{1, \dots, t\}$ , where  $v_{t+1} = v_1$ . When  $t = 2$ , the cycle  $C = v_1v_2v_1$  on two vertices  $v_1, v_2$  coincides with the edge  $v_1v_2$ , and when  $t = 1$ , the cycle  $C = v_1$  coincides with the vertex  $v_1$ . So, all vertices and edges in a graph can be considered as cycles of lengths 1 and 2, respectively. A graph  $G$  is hamiltonian if  $G$  contains a Hamilton cycle, i.e. a cycle of length  $n$ .

Paths and cycles in a graph  $G$  are considered as subgraphs of  $G$ . If  $Q$  is a path or a cycle, then the length of  $Q$ , denoted by  $|Q|$ , is  $|E(Q)|$ . We write  $Q$  with a given orientation by  $\vec{Q}$ . For  $x, y \in V(Q)$ , we denote by  $x\vec{Q}y$  the subpath of  $Q$  in the chosen direction from  $x$  to  $y$ . For  $x \in V(Q)$ , we denote the  $h$ -th successor and the  $h$ -th predecessor of  $x$  on  $\vec{Q}$  by  $x^{+h}$  and  $x^{-h}$ , respectively. We abbreviate  $x^{+1}$  and  $x^{-1}$  by  $x^+$  and  $x^-$ , respectively.

**Special definitions.** Let  $G$  be a graph,  $C$  a longest cycle in  $G$  and  $P = x\vec{P}y$  a longest path in  $G \setminus C$  of length  $\bar{p} \geq 0$ . Let  $\xi_1, \xi_2, \dots, \xi_s$  be the elements of  $N_C(x) \cup N_C(y)$  occuring on  $C$  in a consecutive order and let

$$I_i = \xi_i\vec{C}\xi_{i+1}, \quad I_i^* = \xi_i^+\vec{C}\xi_{i+1}^- \quad (i = 1, 2, \dots, s),$$

where  $\xi_{s+1} = \xi_1$ .

(\*1) We call  $I_1, I_2, \dots, I_s$  elementary segments on  $C$  induced by  $N_C(x) \cup N_C(y)$ .

(\*2) We call a path  $L = z\vec{L}w$  an intermediate path between elementary segments  $I_a$  and  $I_b$  if

$$z \in V(I_a^*), \quad w \in V(I_b^*), \quad V(L) \cap V(C \cup P) = \{z, w\}.$$

(\*3) Denote by  $M(I_{i_1}, I_{i_2}, \dots, I_{i_t})$  the set of all intermediate paths between elementary segments  $I_{i_1}, I_{i_2}, \dots, I_{i_t}$ .

**Lemma 1.** Let  $G$  be a graph,  $C$  a longest cycle in  $G$  and  $P = x\vec{P}y$  a longest path in  $G \setminus C$  of length  $\bar{p} \geq 1$ . If  $|N_C(x)| \geq 2$ ,  $|N_C(y)| \geq 2$  and  $N_C(x) \neq N_C(y)$  then

$$|C| \geq \begin{cases} 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta & \text{if } \bar{p} = 1, \\ \max\{2\bar{p} + 8, 4\delta - 2\bar{p}\} & \text{if } \bar{p} \geq 2, \end{cases}$$

where  $\sigma_1 = |N_C(x) \setminus N_C(y)|$  and  $\sigma_2 = |N_C(y) \setminus N_C(x)|$ .

**Lemma 2.** Let  $G$  be a graph,  $C$  a longest cycle in  $G$  and  $P = x\vec{P}y$  a longest path in  $G \setminus C$  of length  $\bar{p} \geq 0$ . If  $N_C(x) = N_C(y)$  and  $|N_C(x)| \geq 2$  then for each elementary segments  $I_a$  and  $I_b$  induced by  $N_C(x) \cup N_C(y)$ ,

(a1) if  $L$  is an intermediate path between  $I_a$  and  $I_b$  then

$$|I_a| + |I_b| \geq 2\bar{p} + 2|L| + 4,$$

(a2) if  $M(I_a, I_b) \subseteq E(G)$  and  $|M(I_a, I_b)| = i$  ( $i \in \{1, 2, 3\}$ ) then

$$|I_a| + |I_b| \geq 2\bar{p} + i + 5.$$

**Lemma 3.** Let  $G$  be a graph,  $S$  a cut set in  $G$  and  $H$  a connected component of  $G \setminus S$  of order  $h$ . Then

$$q_H \geq \frac{h(2\delta - h + 1)}{2},$$

where  $q_H = |\{xy \in E(G) : \{x, y\} \cap V(H) \neq \emptyset\}|$ .

**Lemma 4.** Let  $G$  be a 2-connected graph. If  $\delta \geq (n - 2)/3$  then either

$$q \geq \begin{cases} 9 & \text{if } \delta = 2, \\ \frac{3(\delta-1)(\delta+2)}{2} & \text{if } \delta \geq 3, \end{cases}$$

or each longest cycle in  $G$  is a dominating cycle.

### 3 Proofs

**Proof of Lemma 1.** Put

$$A_1 = N_C(x) \setminus N_C(y), \quad A_2 = N_C(y) \setminus N_C(x), \quad M = N_C(x) \cap N_C(y).$$

By the hypothesis,  $N_C(x) \neq N_C(y)$ , implying that

$$\max\{|A_1|, |A_2|\} \geq 1.$$

Let  $\xi_1, \xi_2, \dots, \xi_s$  be the elements of  $N_C(x) \cup N_C(y)$  occurring on  $C$  in a consecutive order. Put  $I_i = \xi_i \overrightarrow{C} \xi_{i+1}$  ( $i = 1, 2, \dots, s$ ), where  $\xi_{s+1} = \xi_1$ . Clearly,  $s = |A_1| + |A_2| + |M|$ . Since  $C$  is extreme,  $|I_i| \geq 2$  ( $i = 1, 2, \dots, s$ ). Moreover, if  $\{\xi_i, \xi_{i+1}\} \cap M \neq \emptyset$  for some  $i \in \{1, 2, \dots, s\}$  then  $|I_i| \geq \bar{p} + 2$ . In addition, if either  $\xi_i \in A_1, \xi_{i+1} \in A_2$  or  $\xi_i \in A_2, \xi_{i+1} \in A_1$  then again  $|I_i| \geq \bar{p} + 2$ .

**Case 1.**  $\bar{p} = 1$ .

**Case 1.1.**  $|A_i| \geq 1$  ( $i = 1, 2$ ).

It follows that among  $I_1, I_2, \dots, I_s$  there are  $|M| + 2$  segments of length at least  $\bar{p} + 2$ . Observing also that each of the remaining  $s - (|M| + 2)$  segments has a length at least 2, we get

$$\begin{aligned} |C| &\geq (\bar{p} + 2)(|M| + 2) + 2(s - |M| - 2) \\ &= 3(|M| + 2) + 2(|A_1| + |A_2| - 2) \\ &= 2|A_1| + 2|A_2| + 3|M| + 2. \end{aligned}$$

Since  $|A_1| = d(x) - |M| - 1$  and  $|A_2| = d(y) - |M| - 1$ , we have

$$|C| \geq 2d(x) + 2d(y) - |M| - 2 \geq 3\delta + d(x) - |M| - 2.$$

Recalling that  $d(x) = |M| + |A_1| + 1$ , we get

$$|C| \geq 3\delta + |A_1| - 1 = 3\delta + \sigma_1 - 1.$$

Analogously,  $|C| \geq 3\delta + \sigma_2 - 1$ . So,

$$|C| \geq 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta.$$

**Case 1.2.** Either  $|A_1| \geq 1, |A_2| = 0$  or  $|A_1| = 0, |A_2| \geq 1$ .

Assume w.l.o.g. that  $|A_1| \geq 1$  and  $|A_2| = 0$ , i.e.  $|N_C(y)| = |M| \geq 2$  and  $s = |A_1| + |M|$ . Hence, among  $I_1, I_2, \dots, I_s$  there are  $|M| + 1$  segments of length at least  $\bar{p} + 2 = 3$ . Taking into account that each of the remaining  $s - (|M| + 1)$  segments has a length at least 2 and  $|M| + 1 = d(y)$ , we get

$$\begin{aligned} |C| &\geq 3(|M| + 1) + 2(s - |M| - 1) = 3d(y) + 2(|A_1| - 1) \\ &\geq 3\delta + |A_1| - 1 = 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta. \end{aligned}$$

**Case 2.**  $\bar{p} \geq 2$ .

We first prove that  $|C| \geq 2\bar{p} + 8$ . Since  $|N_C(x)| \geq 2$  and  $|N_C(y)| \geq 2$ , there are at least two segments among  $I_1, I_2, \dots, I_s$  of length at least  $\bar{p} + 2$ . If  $|M| = 0$  then clearly  $s \geq 4$  and

$$|C| \geq 2(\bar{p} + 2) + 2(s - 2) \geq 2\bar{p} + 8.$$

Otherwise, since  $\max\{|A_1|, |A_2|\} \geq 1$ , there are at least three elementary segments of length at least  $\bar{p} + 2$ , i.e.

$$|C| \geq 3(\bar{p} + 2) \geq 2\bar{p} + 8.$$

So, in any case,  $|C| \geq 2\bar{p} + 8$ .

To prove that  $|C| \geq 4\delta - 2\bar{p}$ , we distinguish two main cases.

**Case 2.1.**  $|A_i| \geq 1$  ( $i = 1, 2$ ).

It follows that among  $I_1, I_2, \dots, I_s$  there are  $|M| + 2$  segments of length at least  $\bar{p} + 2$ . Further, since each of the remaining  $s - (|M| + 2)$  segments has a length at least 2, we get

$$\begin{aligned} |C| &\geq (\bar{p} + 2)(|M| + 2) + 2(s - |M| - 2) \\ &= (\bar{p} - 2)|M| + (2\bar{p} + 4|M| + 4) + 2(|A_1| + |A_2| - 2) \\ &\geq 2|A_1| + 2|A_2| + 4|M| + 2\bar{p}. \end{aligned}$$

Observing also that

$$|A_1| + |M| + \bar{p} \geq d(x), \quad |A_2| + |M| + \bar{p} \geq d(y),$$

we have

$$\begin{aligned} &2|A_1| + 2|A_2| + 4|M| + 2\bar{p} \\ &\geq 2d(x) + 2d(y) - 2\bar{p} \geq 4\delta - 2\bar{p}, \end{aligned}$$

implying that  $|C| \geq 4\delta - 2\bar{p}$ .

**Case 2.2.** Either  $|A_1| \geq 1, |A_2| = 0$  or  $|A_1| = 0, |A_2| \geq 1$ .

Assume w.l.o.g. that  $|A_1| \geq 1$  and  $|A_2| = 0$ , i.e.  $|N_C(y)| = |M| \geq 2$  and  $s = |A_1| + |M|$ . It follows that among  $I_1, I_2, \dots, I_s$  there are  $|M| + 1$  segments of length at least  $\bar{p} + 2$ . Observing also that  $|M| + \bar{p} \geq d(y) \geq \delta$ , i.e.  $2\bar{p} + 4|M| \geq 4\delta - 2\bar{p}$ , we get

$$\begin{aligned} |C| &\geq (\bar{p} + 2)(|M| + 1) \geq (\bar{p} - 2)(|M| - 1) + 2\bar{p} + 4|M| \\ &\geq 2\bar{p} + 4|M| \geq 4\delta - 2\bar{p}. \quad \blacksquare \end{aligned}$$

**Proof of Lemma 2.** Let  $\xi_1, \xi_2, \dots, \xi_s$  be the elements of  $N_C(x)$  occurring on  $C$  in a consecutive order. Put  $I_i = \xi_i \overrightarrow{C} \xi_{i+1}$  ( $i = 1, 2, \dots, s$ ), where  $\xi_{s+1} = \xi_1$ . To prove (a1), let  $L = z \overrightarrow{L} w$  be an intermediate path between elementary segments  $I_a$  and  $I_b$  with  $z \in V(I_a^*)$  and  $w \in V(I_b^*)$ . Put

$$\begin{aligned} |\xi_a \overrightarrow{C} z| &= d_1, \quad |z \overrightarrow{C} \xi_{a+1}| = d_2, \quad |\xi_b \overrightarrow{C} w| = d_3, \quad |w \overrightarrow{C} \xi_{b+1}| = d_4, \\ C' &= \xi_a x \overrightarrow{P} y \xi_b \overleftarrow{C} z \overrightarrow{L} w \overrightarrow{C} \xi_a. \end{aligned}$$

Clearly,

$$|C'| = |C| - d_1 - d_3 + |L| + |P| + 2.$$

Since  $C$  is extreme, we have  $|C| \geq |C'|$ , implying that  $d_1 + d_3 \geq \bar{p} + |L| + 2$ . By a symmetric argument,  $d_2 + d_4 \geq \bar{p} + |L| + 2$ . Hence

$$|I_a| + |I_b| = \sum_{i=1}^4 d_i \geq 2\bar{p} + 2|L| + 4.$$

To proof (a2), let  $M(I_a, I_b) \subseteq E(G)$  and  $|M(I_a, I_b)| = i$  ( $i \in \{1, 2, 3\}$ ).

**Case 1.**  $i = 1$ .

It follows that  $M(I_a, I_b)$  consists of a single intermediate edge  $L = zw$ . By (a1),

$$|I_a| + |I_b| \geq 2\bar{p} + 2|L| + 4 = 2\bar{p} + 6.$$

**Case 2.**  $i = 2$ .

It follows that  $M(I_a, I_b)$  consists of two edges  $e_1, e_2$ . Put  $e_1 = z_1 w_1$  and  $e_2 = z_2 w_2$ , where  $\{z_1, z_2\} \subseteq V(I_a^*)$  and  $\{w_1, w_2\} \subseteq V(I_b^*)$ .

**Case 2.1.**  $z_1 \neq z_2$  and  $w_1 \neq w_2$ .

Assume w.l.o.g. that  $z_1$  and  $z_2$  occur in this order on  $I_a$ .

**Case 2.1.1.**  $w_2$  and  $w_1$  occur in this order on  $I_b$ .

Put

$$\begin{aligned} |\xi_a \overrightarrow{C} z_1| &= d_1, \quad |z_1 \overrightarrow{C} z_2| = d_2, \quad |z_2 \overrightarrow{C} \xi_{a+1}| = d_3, \\ |\xi_b \overrightarrow{C} w_2| &= d_4, \quad |w_2 \overrightarrow{C} w_1| = d_5, \quad |w_1 \overrightarrow{C} \xi_{b+1}| = d_6, \\ C' &= \xi_a \overrightarrow{C} z_1 w_1 \overleftarrow{C} w_2 z_2 \overrightarrow{C} \xi_b x \overrightarrow{P} y \xi_{b+1} \overrightarrow{C} \xi_a. \end{aligned}$$

Clearly,

$$\begin{aligned} |C'| &= |C| - d_2 - d_4 - d_6 + |\{e_1\}| + |\{e_2\}| + |P| + 2 \\ &= |C| - d_2 - d_4 - d_6 + \bar{p} + 4. \end{aligned}$$

Since  $C$  is extreme,  $|C| \geq |C'|$ , implying that  $d_2 + d_4 + d_6 \geq \bar{p} + 4$ . By a symmetric argument,  $d_1 + d_3 + d_5 \geq \bar{p} + 4$ . Hence

$$|I_a| + |I_b| = \sum_{i=1}^6 d_i \geq 2\bar{p} + 8.$$

**Case 2.1.2.**  $w_1$  and  $w_2$  occur in this order on  $I_b$

Putting

$$C' = \xi_a \vec{C}_{z_1 w_1} \vec{C}_{w_2 z_2} \vec{C}_{\xi_b x} \vec{P}_{y \xi_{b+1}} \vec{C}_{\xi_a},$$

we can argue as in Case 2.1.1.

**Case 2.2.** Either  $z_1 = z_2$ ,  $w_1 \neq w_2$  or  $z_1 \neq z_2$ ,  $w_1 = w_2$ .

Assume w.l.o.g. that  $z_1 \neq z_2$ ,  $w_1 = w_2$  and  $z_1, z_2$  occur in this order on  $I_a$ .

Put

$$\begin{aligned} |\xi_a \vec{C}_{z_1}| &= d_1, \quad |z_1 \vec{C}_{z_2}| = d_2, \quad |z_2 \vec{C}_{\xi_{a+1}}| = d_3, \\ |\xi_b \vec{C}_{w_1}| &= d_4, \quad |w_1 \vec{C}_{\xi_{b+1}}| = d_5, \\ C' &= \xi_a x \vec{P}_{y \xi_b} \vec{C}_{z_1 w_1} \vec{C}_{\xi_a}, \\ C'' &= \xi_a \vec{C}_{z_2 w_1} \vec{C}_{\xi_{a+1} x} \vec{P}_{y \xi_{b+1}} \vec{C}_{\xi_a}. \end{aligned}$$

Clearly,

$$|C'| = |C| - d_1 - d_4 + |\{e_1\}| + |P| + 2 = |C| - d_1 - d_4 + \bar{p} + 3,$$

$$|C''| = |C| - d_3 - d_5 + |\{e_2\}| + |P| + 2 = |C| - d_3 - d_5 + \bar{p} + 3.$$

Since  $C$  is extreme,  $|C| \geq |C'|$  and  $|C| \geq |C''|$ , implying that

$$d_1 + d_4 \geq \bar{p} + 3, \quad d_3 + d_5 \geq \bar{p} + 3.$$

Hence,

$$|I_a| + |I_b| = \sum_{i=1}^5 d_i \geq d_1 + d_3 + d_4 + d_5 + 1 \geq 2\bar{p} + 7.$$

**Case 3.**  $i = 3$ .

It follows that  $M(I_a, I_b)$  consists of three edges  $e_1, e_2, e_3$ . Let  $e_i = z_i w_i$  ( $i = 1, 2, 3$ ), where  $\{z_1, z_2, z_3\} \subseteq V(I_a^*)$  and  $\{w_1, w_2, w_3\} \subseteq V(I_b^*)$ . If there are two independent edges among  $e_1, e_2, e_3$  then we can argue as in Case 2.1. Otherwise, we can assume w.l.o.g. that  $w_1 = w_2 = w_3$  and  $z_1, z_2, z_3$  occur in this order on  $I_a$ . Put

$$|\xi_a \vec{C}_{z_1}| = d_1, \quad |z_1 \vec{C}_{z_2}| = d_2, \quad |z_2 \vec{C}_{z_3}| = d_3,$$

$$|z_3 \vec{C}_{\xi_{a+1}}| = d_4, \quad |\xi_b \vec{C}_{w_1}| = d_5, \quad |w_1 \vec{C}_{\xi_{b+1}}| = d_6,$$

$$\begin{aligned} C' &= \xi_a x \vec{P}_{y\xi_b} \overleftarrow{C}_{z_1 w_1} \vec{C}_{\xi_a}, \\ C'' &= \xi_a \vec{C}_{z_3 w_1} \overleftarrow{C}_{\xi_{a+1} x} \vec{P}_{y\xi_{b+1}} \vec{C}_{\xi_a}. \end{aligned}$$

Clearly,

$$\begin{aligned} |C'| &= |C| - d_1 - d_5 + |\{e_1\}| + \bar{p} + 2, \\ |C''| &= |C| - d_4 - d_6 + |\{e_3\}| + \bar{p} + 2. \end{aligned}$$

Since  $C$  is extreme, we have  $|C| \geq |C'|$  and  $|C| \geq |C''|$ , implying that

$$d_1 + d_5 \geq \bar{p} + 3, \quad d_4 + d_6 \geq \bar{p} + 3.$$

Hence,

$$|I_a| + |I_b| = \sum_{i=1}^6 d_i \geq d_1 + d_4 + d_5 + d_6 + 2 \geq 2\bar{p} + 8. \quad \blacksquare$$

**Proof of Lemma 3.** Put

$$V(H) = \{v_1, \dots, v_h\}, \quad |N(v_i) \cap S| = \beta_i \quad (i = 1, \dots, h).$$

Observing that  $h \geq d(v_i) - \beta_i + 1 \geq \delta - \beta_i + 1$  for each  $i \in \{1, 2, \dots, h\}$ , we have  $\beta_i \geq \delta - h + 1$  ( $i = 1, 2, \dots, h$ ). Therefore,

$$\begin{aligned} q_H &= q(H) + \sum_{i=1}^h \beta_i = \frac{1}{2} \sum_{i=1}^h d_H(v_i) + \sum_{i=1}^h \beta_i \\ &= \frac{1}{2} \sum_{i=1}^h (d_H(v_i) + \beta_i) + \frac{1}{2} \sum_{i=1}^h \beta_i = \frac{1}{2} \sum_{i=1}^h d(v_i) + \frac{1}{2} \sum_{i=1}^h (\delta - h + 1) \\ &\geq \frac{1}{2} h\delta + \frac{1}{2} h(\delta - h + 1) = \frac{h(2\delta - h + 1)}{2}. \quad \blacksquare \end{aligned}$$

**Proof of Lemma 4.** Let  $C$  be a longest cycle in  $G$  and  $P = x_1 \vec{P} x_2$  a longest path in  $G \setminus C$  of length  $\bar{p}$ . If  $|V(P)| \leq 1$  then  $C$  is a dominating cycle and we are done. Let  $|V(P)| \geq 2$ , that is  $\bar{p} \geq 1$ . By the hypothesis,

$$|C| + \bar{p} + 1 \leq n \leq 3\delta + 2. \quad (1)$$

Let  $\xi_1, \xi_2, \dots, \xi_s$  be the elements of  $N_C(x_1) \cup N_C(x_2)$  occuring on  $C$  in a consecutive order. Put

$$I_i = \xi_i \vec{C}_{\xi_{i+1}}, \quad I_i^* = \xi_i^+ \vec{C}_{\xi_{i+1}}^- \quad (i = 1, 2, \dots, s),$$

where  $\xi_{s+1} = \xi_1$ . Let  $Q$  be a longest path in  $G$  with  $Q = \xi \vec{Q} \eta$  and  $V(Q) \cap V(C) = \{\xi, \eta\}$ . Since  $C$  is extreme, we have  $|\xi \vec{C} \eta| \geq |Q|$  and  $|\eta \vec{C} \xi| \geq |Q|$ , implying that

$$|C| = |\xi \vec{C} \eta| + |\eta \vec{C} \xi| \geq 2|Q|. \quad (2)$$



**Case 1.**  $\delta = 2$ .

Since  $\kappa \geq 2$  and  $\bar{p} \geq 1$ , we have  $|Q| \geq 3$ . By (2),

$$|C| = |y\vec{C}z| + |z\vec{C}y| \geq 2|Q| \geq 6,$$

implying that  $q \geq |C| + |Q| \geq 9$ .

**Case 2.**  $\delta = 3$ .

If  $n \geq 10$  then

$$q \geq \frac{n\delta}{2} \geq 15 = \frac{3(\delta-1)(\delta+2)}{2}.$$

Let

$$n \leq 9. \quad (3)$$

**Case 2.1.**  $\bar{p} = 1$ .

By (1) and (3),

$$|C| \leq n - \bar{p} - 1 \leq 7. \quad (4)$$

Since  $\bar{p} = 1$  and  $\delta = 3$ , we have  $|N_C(x_i)| \geq 2$  ( $i = 1, 2$ ). If  $N_C(x_1) \neq N_C(x_2)$  then by Lemma 1,  $|C| \geq 3\delta = 9$ , contradicting (4). Let  $N_C(x_1) = N_C(x_2)$ . Further, since  $C$  is extreme and  $\bar{p} = 1$ , we have  $|I_i| \geq 3$  ( $i = 1, 2, \dots, s$ ). If  $s \geq 3$  then  $|C| = \sum_{i=1}^s |I_i| \geq 3s \geq 9$ , contradicting (4). Let  $s = 2$ . If  $M(I_1, I_2) \neq \emptyset$  then by Lemma 2,  $|C| = |I_1| + |I_2| \geq 2\bar{p} + 6 = 8$ , contradicting (4). Thus,  $M(I_1, I_2) = \emptyset$ , implying that  $G \setminus \{\xi_1, \xi_2\}$  is disconnected. Let  $H_1, H_2, \dots, H_t$  be the connected components of  $G \setminus \{\xi_1, \xi_2\}$ . Clearly,  $t \geq 3$ . Put

$$h_i = |V(H_i)|, \quad q_i = |\{xy \in E(G) : \{x, y\} \cap V(H_i) \neq \emptyset\}| \quad (i = 1, 2, \dots, t). \quad (5)$$

Assume w.l.o.g. that  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2$ ) and  $V(H_3) = \{x_1, x_2\}$ . It means that  $h_i \geq 2$  ( $i = 1, 2, 3$ ). If  $h_i \geq 4$  for some  $i \in \{1, 2\}$  then

$$|C| \geq h_1 + h_2 + |\{\xi_1, \xi_2\}| \geq 8,$$

contradicting (4). Let  $2 \leq h_i \leq 3$  ( $i = 1, 2, 3$ ). By Lemma 3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{h_i(7 - h_i)}{2} \geq 5 \quad (i = 1, 2, 3).$$

Hence

$$q \geq \sum_{i=1}^3 q_i \geq 15 = \frac{3(\delta-1)(\delta+2)}{2}.$$

**Case 2.2.**  $\bar{p} \geq 2$ .

By (1) and (3),  $|C| \leq n - \bar{p} - 1 \leq 6$ .

**Case 2.2.1.** There is a cycle in  $G \setminus C$ .

Let  $C'$  be a cycle in  $G \setminus C$ . Since  $\kappa \geq 2$ , there are two disjoint paths connecting  $C'$  and  $C$ , implying that  $|Q| \geq 4$ . By (2),  $|C| \geq 2|Q| \geq 8$ , contradicting (4).

**Case 2.2.2.**  $G \setminus C$  is acyclic.

It follows that

$$|N_C(x_i)| \geq |N(x_i)| - 1 \geq \delta - 1 = 2 \quad (i = 1, 2).$$

Hence  $|Q| \geq \bar{p} + 2 \geq 4$ . By (2),  $|C| \geq 2|Q| \geq 8$ , contradicting (4).

**Case 3.**  $\delta = 4$ .

If  $n \geq 14$  then

$$q \geq \frac{n\delta}{2} \geq 28 > \frac{3(\delta-1)(\delta+2)}{2}.$$

Let

$$n \leq 13. \quad (6)$$

**Case 3.1.**  $\bar{p} = 1$ .

By (1) and (6),

$$|C| \leq n - \bar{p} - 1 \leq 11. \quad (7)$$

Since  $\bar{p} = 1$  and  $\delta = 4$ , we have  $|N_C(x_i)| \geq 3$  ( $i = 1, 2$ ). If  $N_C(x_1) \neq N_C(x_2)$  then by Lemma 1,  $|C| \geq 3\delta = 12$ , contradicting (7). Let  $N_C(x_1) = N_C(x_2)$ . Further, since  $C$  is extreme and  $\bar{p} = 1$ , we have  $|I_i| \geq 3$  ( $i = 1, \dots, s$ ). If  $s \geq 4$  then  $|C| \geq 3s \geq 12$ , contradicting (7). Thus  $s = 3$ .

**Case 3.1.1.**  $M(I_1, I_2, I_3) = \emptyset$ .

It follows that  $G \setminus \{\xi_1, \xi_2, \xi_3\}$  is disconnected. Let  $H_1, H_2, \dots, H_t$  be the connected components of  $G \setminus \{\xi_1, \xi_2, \xi_3\}$ . Clearly,  $t \geq 4$ . Assume w.l.o.g. that  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2, 3$ ) and  $V(H_4) = \{x_1, x_2\}$ . Using notation (5), we have  $h_i \geq 2$  ( $i = 1, 2, 3$ ) and  $h_4 = 2$ . If  $h_i \geq 5$  for some  $i \in \{1, 2, 3\}$  then clearly

$$|C| \geq \sum_{i=1}^3 h_i + |\{\xi_1, \xi_2, \xi_3\}| \geq 12,$$

contradicting (7). Let  $2 \leq h_i \leq 4$  ( $i = 1, 2, 3$ ). By Lemma 3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{h_i(9 - h_i)}{2} \geq 7 \quad (i = 1, 2, 3, 4).$$

So,

$$q \geq \sum_{i=1}^4 q_i \geq 28 > \frac{3(\delta-1)(\delta+2)}{2}.$$

**Case 3.1.2.**  $M(I_1, I_2, I_3) \neq \emptyset$ .

Assume w.l.o.g. that  $M(I_1, I_2) \neq \emptyset$ , i.e. there is an intermediate path  $L$  between  $I_1$  and  $I_2$ . By Lemma 2,

$$|I_1| + |I_2| \geq 2\bar{p} + 2|L| + 4 = 2|L| + 6.$$

If  $|L| \geq 2$  then  $|I_1| + |I_2| \geq 10$  and hence  $|C| = |I_1| + |I_2| + |I_3| \geq 13$ , contradicting (7). Otherwise,  $|L| = 1$ , implying that  $M(I_1, I_2, I_3) \subseteq E(G)$ . If  $|M(I_1, I_2)| \geq 2$

then by Lemma 2,  $|I_1| + |I_2| \geq 2\bar{p} + 7 = 9$  and  $|C| = \sum_{i=1}^3 |I_i| \geq 12$ , contradicting (7). So,  $|M(I_1, I_2)| = 1$ . By Lemma 2,  $|I_1| + |I_2| \geq 2\bar{p} + 6 = 8$ . Since  $|I_3| \geq 3$ , we have  $|C| = \sum_{i=1}^3 |I_i| \geq 11$ . By (1),  $n \geq |C| + \bar{p} + 1 \geq 13$ . Combining  $n \geq 13$  and  $|C| \geq 11$  with (6) and (7), we get

$$n = 13, |C| = 11, |I_1| + |I_2| = 8, |I_3| = 3, V(G) = V(C \cup P). \quad (8)$$

Since  $|I_1| + |I_2| = 8$  and  $|I_i| \geq 3$  ( $i = 1, 2$ ), we can assume w.l.o.g. that either  $|I_1| = |I_2| = 4$  or  $|I_1| = 3, |I_2| = 5$ . If  $|I_1| = |I_2| = 4$  then by Lemma 2,  $M(I_1, I_3) = M(I_2, I_3) = \emptyset$ , implying that  $|M(I_1, I_2, I_3)| = 1$ . Further, if  $|I_1| = 3$  and  $|I_2| = 5$  then by Lemma 2,  $M(I_1, I_3) = \emptyset$  and  $|M(I_2, I_3)| \leq 1$ , implying that  $|M(I_1, I_2, I_3)| \leq 2$ . So, in any case,

$$1 \leq |M(I_1, I_2, I_3)| \leq 2. \quad (9)$$

Let  $e \in M(I_1, I_2, I_3)$  and  $e = zw$ . Put  $G' = G \setminus e$ . Form a graph  $G''$  in the following way. If  $d(z) \geq \delta = 4$  and  $d(w) \geq \delta = 4$  in  $G'$  then we take  $G'' = G'$ . Next, we let  $d(z) = \delta - 1 = 3$  and  $d(w) \geq \delta = 4$  in  $G'$ . If  $\{\xi_1, \xi_2, \xi_3\} \subseteq N(z)$  then clearly  $d(z) \geq 4$  in  $G'$ , contradicting the hypothesis. Otherwise,  $zv \notin E(G')$  for some  $v \in \{\xi_1, \xi_2, \xi_3\}$  and we take  $G'' = G' + \{zv\}$ . Finally, if  $d(z) = d(w) = 3$  then as above,  $zv \notin E(G')$  and  $wu \notin E(G')$  for some  $v, u \in \{\xi_1, \xi_2, \xi_3\}$  and we take  $G'' = G' + \{zv, wu\}$ . Clearly,  $\delta(G'') = \delta(G) = 4$  and  $q = q(G) \geq q(G'') - 1$ . Furthermore, deleting step by step all edges from  $M(I_1, I_2, I_3)$  and adding at most two appropriate new edges against each deleting edge, we can form a graph  $G^*$  with  $\delta(G^*) = \delta(G) = 4$  and  $q(G) \geq q(G^*) - |M(I_1, I_2, I_3)|$ . By (9),  $q(G) \geq q(G^*) - 2$ . In fact,  $G^* = (G \setminus M(I_1, I_2, I_3)) + E^*$ , where  $E^*$  consists of at most  $2|M(I_1, I_2, I_3)|$  appropriate new edges having exactly one end in common with  $\{\xi_1, \xi_2, \xi_3\}$ , implying that  $G^* \setminus \{\xi_1, \xi_2, \xi_3\}$  is disconnected. Let  $H_1, H_2, H_3, H_4$  be the connected components of  $G^* \setminus \{\xi_1, \xi_2, \xi_3\}$  with  $V(H_i) = V(I_i^*)$  ( $i = 1, 2, 3$ ) and  $V(H_4) = \{x_1, x_2\}$ . Put

$$h_i = |V(H_i)|, \quad q_i = |\{xy \in E(G^*) : \{x, y\} \cap V(H_i) \neq \emptyset\}| \quad (i = 1, 2, 3, 4).$$

Since  $|I_1| + |I_2| = 8$ , we have either  $|I_1| \geq 4$  or  $|I_2| \geq 4$ . Assume w.l.o.g. that  $|I_1| \geq 4$ , that is  $h_1 \geq 3$ . As in Case 3.1.1,  $2 \leq h_i \leq 4$  ( $i = 1, 2, 3, 4$ ). By Lemma 3,

$$\begin{aligned} q_1(G^*) &\geq \frac{h_1(2\delta - h_1 + 1)}{2} = \frac{h_1(9 - h_1)}{2} \geq 9, \\ q_i(G^*) &\geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{h_i(9 - h_i)}{2} \geq 7 \quad (i = 2, 3, 4). \end{aligned}$$

Hence  $q(G^*) \geq \sum_{i=1}^4 q_i(G^*) \geq 30$ , implying that

$$q(G) \geq q(G^*) - 2 \geq 28 > \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 3.2.**  $\bar{p} = 2$ .

Put  $P = x_1x_3x_2$ . By (1) and (6),

$$|C| \leq n - \bar{p} - 1 \leq 10. \quad (10)$$

Since  $\delta = 4$  and  $\bar{p} = 2$ , we have  $|N_C(x_i)| \geq 2$  ( $i = 1, 2$ ). If  $N_C(x_1) \neq N_C(x_2)$  then by Lemma 1,  $|C| \geq 4\delta - 2\bar{p} = 12$ , contradicting (10). Let  $N_C(x_1) = N_C(x_2)$ . Recalling that  $C$  is extreme and  $\bar{p} = 2$ , we conclude that  $|I_i| \geq 4$  ( $i = 1, 2, \dots, s$ ). If  $s \geq 3$  then  $|C| \geq 4s \geq 12$ , contradicting (10). Let  $s = 2$ , implying that  $x_1x_2 \in E(G)$ . By symmetric arguments, we can state that  $N_C(x_1) = N_C(x_2) = N_C(x_3)$ .

**Case 3.2.1.**  $M(I_1, I_2) = \emptyset$ .

Let  $H_1, H_2, \dots, H_t$  be the connected components of  $G \setminus \{\xi_1, \xi_2\}$ . Assume w.l.o.g. that  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2$ ) and  $V(H_3) = \{x_1, x_2, x_3\}$ . Using notation (5), we have  $h_i \geq 3$  ( $i = 1, 2, 3$ ). If  $h_i \geq 6$  for some  $i \in \{1, 2\}$ , then  $|C| \geq h_1 + h_2 + |\{\xi_1, \xi_2\}| \geq 11$ , contradicting (10). Let  $3 \leq h_i \leq 5$  ( $i = 1, 2, 3$ ). By Lemma 3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{h_i(9 - h_i)}{2} \geq 9 \quad (i = 1, 2, 3).$$

Hence

$$q \geq \sum_{i=1}^3 q_i \geq 27 = \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 3.2.2.**  $M(I_1, I_2) \neq \emptyset$ .

By the definition, there is an intermediate path  $L$  between  $I_1$  and  $I_2$ . By Lemma 2,

$$|I_1| + |I_2| \geq 2\bar{p} + 2|L| + 4 = 2|L| + 8.$$

If  $|L| \geq 2$  then  $|C| = |I_1| + |I_2| \geq 12$ , contradicting (10). Otherwise,  $|L| = 1$  and therefore,  $M(I_1, I_2) \subseteq E(G)$ . If  $|M(I_1, I_2)| \geq 2$  then by Lemma 2,

$$|C| = |I_1| + |I_2| \geq 2\bar{p} + 7 = 11,$$

contradicting (10). Now let  $|M(I_1, I_2)| = 1$ , i.e.  $M(I_1, I_2)$  consists of a single edge  $e$ . By Lemma 2,

$$|C| = |I_1| + |I_2| \geq 2\bar{p} + 6 = 10,$$

and by (1),  $n \geq |C| + \bar{p} + 1 \geq 13$ . Combining  $n \geq 13$  and  $|C| \geq 10$  with (6) and (10), we get

$$|C| = |I_1| + |I_2| = 10, \quad n = 13, \quad V(G) = V(C \cup P). \quad (11)$$

Put  $G' = G \setminus e$  and let  $H_1, H_2, H_3$  be the connected components of  $G' \setminus \{\xi_1, \xi_2\}$  with  $V(H_i) = V(I_i^*)$  ( $i = 1, 2$ ) and  $V(H_3) = V(P)$ . Since  $|I_1| + |I_2| = 10$ , we can assume w.l.o.g. that  $|I_1| \geq 5$ . Using notation (5) for  $G'$ , we have  $h_1 \geq |I_1| - 1 \geq 4$  and  $h_i \geq 3$  ( $i = 2, 3$ ). If  $h_i \geq 6$  for some  $i \in \{1, 2\}$  then

$|C| \geq h_1 + h_2 + |\{\xi_1, \xi_2\}| \geq 11$ , contradicting (11). Let  $4 \leq h_1 \leq 5$  and  $3 \leq h_i \leq 5$  ( $i = 2, 3$ ). If  $\delta(G') = \delta(G)$  then we can argue as in Case 3.2.1. Otherwise, as in Case 3.1.2, we can form a graph  $G^*$  by adding at most two new edges in  $G'$  such that  $\delta(G^*) = \delta(G)$  and  $G^* \setminus \{\xi_1, \xi_2\}$  has exactly three connected components. Recalling that  $4 \leq h_1 \leq 5$  and using Lemma 3, we get

$$q_1(G^*) \geq \frac{h_1(2\delta - h_1 + 1)}{2} = \frac{h_1(9 - h_1)}{2} = 10,$$

$$q_i(G^*) \geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{h_i(9 - h_i)}{2} \geq 9 \quad (i = 2, 3).$$

So,  $q(G^*) \geq \sum_{i=1}^3 q_i(G^*) \geq 28$ , implying that

$$q(G) \geq (G^*) - 1 \geq 27 = \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 3.3.**  $\bar{p} = 3$ .

By (1) and (6),

$$|C| \leq n - \bar{p} - 1 \leq 9. \quad (12)$$

Since  $\delta = 4$  and  $\bar{p} = 3$ , we have  $|N_C(x_i)| \geq 1$  ( $i = 1, 2$ ). If  $|N_C(x_i)| \geq 2$  for some  $i \in \{1, 2\}$  then  $|Q| \geq \bar{p} + 2 = 5$  and by (2),  $|C| \geq 2|Q| \geq 10$ , contradicting (12). Let  $|N_C(x_i)| = 1$  ( $i = 1, 2$ ). If  $N_C(x_1) \neq N_C(x_2)$  then again  $|Q| \geq 5$  and  $|C| \geq 10$ , contradicting (12). Thus,  $N_C(x_1) = N_C(x_2)$ . It follows that  $G[V(P)]$  is complete. Since  $\kappa \geq 2$ , there are two disjoint paths connecting  $G[V(P)]$  and  $C$ , implying that  $|Q| \geq 5$  and  $|C| \geq 10$ , contradicting (12).

**Case 3.4.**  $\bar{p} = 4$ .

Put  $P = x_1x_3x_4x_5x_2$ . By (1) and (6),

$$|C| \leq n - \bar{p} - 1 \leq 8. \quad (13)$$

**Case 3.4.1.**  $x_1x_2 \in E(G)$ .

Put  $C' = x_1x_2x_5x_4x_3x_1$ . Since  $\kappa \geq 2$ , there are two disjoint paths connecting  $C'$  and  $C$ . Since  $|C'| = 5$ , we have  $|Q| \geq 5$  and by (2),  $|C| \geq 2|Q| \geq 10$ , contradicting (13).

**Case 3.4.2.**  $x_1x_2 \notin E(G)$ .

As in Case 3.3, it can be shown that

$$N_C(x_1) = N_C(x_2), \quad |N_C(x_1)| = |N_C(x_2)| = 1.$$

Since  $\delta = 4$ , we have  $\{x_1x_4, x_1x_5, x_2x_3, x_2x_4\} \subset E(G)$ . Hence,  $x_1x_4x_5x_2x_3x_1$  is a Hamilton cycle in  $G[V(P)]$  and we can argue as in Case 3.4.1.

**Case 3.5.**  $\bar{p} \geq 5$ .

If  $|C| = n$  then  $C$  is a dominating cycle. Otherwise, by Theorem D,  $|C| \geq 2\delta = 8$ . On the other hand, by (1) and (6),  $|C| \leq n - \bar{p} - 1 \leq 7$ , a

contradiction.

**Case 4.**  $\delta \geq 5$ .

If  $C$  is a Hamilton cycle then we are done. Otherwise, by Theorem D,

$$c \geq 2\delta. \quad (14)$$

By (1),  $\bar{p} \leq 3\delta - |C| + 1 \leq \delta + 1$ . So,

$$1 \leq \bar{p} \leq \delta + 1. \quad (15)$$

We distinguish two main cases, namely  $1 \leq \bar{p} \leq \delta - 3$  and  $\delta - 2 \leq \bar{p} \leq \delta + 1$ .

**Case 4.1.**  $1 \leq \bar{p} \leq \delta - 3$ .

It follows that

$$(\bar{p} + 2)(\delta - \bar{p}) = (\bar{p} - 1)(\delta - \bar{p} - 3) + 3\delta - 3 \geq 3\delta - 3.$$

By Theorem E,

$$|C| \geq 3\delta - 3. \quad (16)$$

By (1) and (16),  $\bar{p} \leq 3\delta - |C| + 1 \leq 4$ . So,

$$1 \leq \bar{p} \leq 4. \quad (17)$$

**Case 4.1.1.**  $\bar{p} = 1$ .

It follows that  $|N_C(x_i)| \geq \delta - 1 > 2$  ( $i = 1, 2$ ). By (1),  $|C| \leq 3\delta + 1 - \bar{p} = 3\delta$ . Combining this with (16), we have

$$3\delta - 3 \leq |C| \leq 3\delta.$$

**Case 4.1.1.1.**  $3\delta - 3 \leq |C| \leq 3\delta - 1$ .

If  $N_C(x_1) \neq N_C(x_2)$  then by Lemma 1,  $|C| \geq 3\delta$ , contradicting the hypothesis. Let  $N_C(x_1) = N_C(x_2)$ . Since  $C$  is extreme and  $\bar{p} = 1$ , we have  $|I_i| \geq 3$  ( $i = 1, \dots, s$ ). If  $s \geq \delta$  then  $|C| \geq 3s \geq 3\delta$ , again contradicting the hypothesis. Let  $s \leq \delta - 1$ . On the other hand,  $s = |N_C(x_1)| = d(x_1) - 1 \geq \delta - 1$ , implying that  $s = \delta - 1$ .

**Claim 1.**  $M(I_1, I_2, \dots, I_s) \subseteq E(G)$  and  $|M(I_1, I_2, \dots, I_s)| \leq \delta - 2$ .

Assume first that  $3\delta - 3 \leq |C| \leq 3\delta - 2$ . If  $M(I_a, I_b) \neq \emptyset$  for some two elementary segments  $I_a$  and  $I_b$  then by Lemma 2,  $|I_a| + |I_b| \geq 2\bar{p} + 6 = 8$ , implying that  $|C| \geq 3\delta - 1$ , a contradiction. Otherwise,  $|M(I_1, I_2, \dots, I_s)| = 0 < \delta - 2$ . Now let  $|C| = 3\delta - 1$ . If  $M(I_1, I_2, \dots, I_s) = \emptyset$  then we are done. Let  $M(I_1, I_2, \dots, I_s) \neq \emptyset$ , i.e.  $M(I_a, I_b) \neq \emptyset$  for some elementary segments  $I_a$  and  $I_b$ . By the definition, there is an intermediate path  $L$  between  $I_a$  and  $I_b$ . If  $|L| \geq 2$  then by Lemma 2,  $|I_a| + |I_b| \geq 2\bar{p} + 2|L| + 4 = 10$ , implying that  $|C| \geq 3\delta$ , a contradiction. Otherwise,  $M(I_1, I_2, \dots, I_s) \subseteq E(G)$  and by Lemma

2,  $|I_a| + |I_b| \geq 2\bar{p} + 6 = 8$ , i.e.  $|C| \geq 3\delta - 1$ . Recalling that  $|C| = 3\delta - 1$ , we can state that

$$|I_a| + |I_b| = 8 \text{ and } |I_i| = 3 \text{ for each } i \in \{1, 2, \dots, s\} \setminus \{a, b\}.$$

If  $|I_a| = |I_b| = 4$  then by Lemma 2,  $M(I_i, I_j) = \emptyset$  if  $\{i, j\} \neq \{a, b\}$ , i.e.  $|M(I_1, I_2, \dots, I_s)| = 1 < \delta - 2$ . Otherwise, assume w.l.o.g. that  $|I_a| = 5$  and  $|I_b| = 3$ , i.e.  $|I_a| = 5$  and  $|I_i| = 3$  for each  $i \in \{1, 2, \dots, s\} \setminus \{a\}$ . As above,  $|M(I_a, I_i)| \leq 1$  for each  $i \in \{1, 2, \dots, s\} \setminus \{a\}$ . Observing also that  $M(I_i, I_j) = \emptyset$  for each distinct  $i, j$  if  $a \notin \{i, j\}$ , we conclude that  $|M(I_1, I_2, \dots, I_s)| \leq s - 1 = \delta - 2$ . Claim 1 is proved.  $\Delta$

Put  $G' = G \setminus M(I_1, I_2, \dots, I_s)$ . As in Case 3.1.2, we can form a graph  $G^*$  by adding at most  $2|M(I_1, I_2, \dots, I_s)|$  new edges in  $G'$  such that  $\delta(G^*) = \delta(G)$ ,  $G^* \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  is disconnected and  $q(G) \geq q(G^*) - |M(I_1, I_2, \dots, I_s)|$ . By Claim 1,

$$q(G) \geq q(G^*) - \delta + 2. \quad (18)$$

Let  $H_1, H_2, \dots, H_{s+1}$  be the connected components of  $G^* \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  with  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2, \dots, s$ ) and  $V(H_{s+1}) = \{x_1, x_2\}$ . Using notation (5) for  $G^*$ , we have  $h_i \geq 2$  ( $i = 1, 2, \dots, s+1$ ). If  $h_i \geq 6$  for some  $i \in \{1, 2, \dots, s\}$  then  $n \geq 3\delta + 3$ , contradicting (1). Let  $2 \leq h_i \leq 5 < 2\delta - 1$  ( $i = 1, 2, \dots, s+1$ ). It follows that  $(h_i - 2)(2\delta - h_i - 1) \geq 0$  which is equivalent to

$$\frac{h_i(2\delta - h_i + 1)}{2} \geq 2\delta - 1 \quad (i = 1, 2, \dots, s+1).$$

By Lemma 3,  $q_i(G^*) \geq 2\delta - 1$  ( $i = 1, 2, \dots, s+1$ ), implying that

$$q(G^*) \geq \sum_{i=1}^{s+1} q_i(G^*) \geq (s+1)(2\delta - 1) = \delta(2\delta - 1).$$

By (18),

$$q \geq (G^*) - \delta + 2 \geq 2(\delta^2 - \delta + 1) \geq \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 4.1.1.2.**  $|C| = 3\delta$ .

**Case 4.1.1.2.1.**  $N_C(x_1) \neq N_C(x_2)$ .

It follows that  $\max\{\sigma_1, \sigma_2\} \geq 1$ , where

$$\sigma_1 = |N_C(x_1) \setminus N_C(x_2)|, \quad \sigma_2 = |N_C(x_2) \setminus N_C(x_1)|.$$

If  $\max\{\sigma_1, \sigma_2\} \geq 2$  then by Lemma 1,  $|C| \geq 3\delta + 1$ , contradicting the hypothesis. Let  $\max\{\sigma_1, \sigma_2\} = 1$ . Clearly  $s \geq \delta$  and  $|I_i| \geq 3$  ( $i = 1, 2, \dots, s$ ). If  $s \geq \delta + 1$  then  $|C| \geq 3s \geq 3\delta + 3$ , a contradiction. Let  $s = \delta$ , implying that  $|I_i| = 3$  ( $i = 1, 2, \dots, s$ ). By Lemma 2,  $M(I_1, I_2, \dots, I_s) = \emptyset$ . Let  $H_1, H_2, \dots, H_{s+1}$  be the connected components of  $G \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  with  $V(H_i) = V(I_i^*)$  ( $i = 1, 2, \dots, s$ )

and  $V(H_{s+1}) = \{x_1, x_2\}$ . Using notation (5), we have  $h_i = 2$  ( $i = 1, 2, \dots, s+1$ ). By Lemma 3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = 2\delta - 1 \quad (i = 1, 2, \dots, s+1),$$

implying that

$$q \geq \sum_{i=1}^{s+1} q_i \geq (s+1)(2\delta - 1) = (\delta + 1)(2\delta - 1) > \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 4.1.1.2.2.**  $N_C(x_1) = N_C(x_2)$ .

Clearly,  $s \geq \delta - 1$ . If  $s \geq \delta$  then we can argue as in Case 4.1.1.2.1. Let  $s = \delta - 1$ . If  $|I_i| + |I_j| \geq 10$  for some distinct  $i, j \in \{1, 2, \dots, s\}$  then  $|C| \geq 10 + 3(s - 2) = 3\delta + 1$ , contradicting the hypothesis. Hence

$$|I_i| + |I_j| \leq 9 \quad \text{for each distinct } i, j \in \{1, 2, \dots, s\}. \quad (19)$$

**Claim 2.**  $M(I_1, I_2, \dots, I_s) \subseteq E(G)$  and

- (\*1) if  $\max_i |I_i| \leq 4$  then  $|M(I_1, I_2, \dots, I_s)| \leq 3$ ,
- (\*2) if  $\max_i |I_i| = 5$  then  $|M(I_1, I_2, \dots, I_s)| \leq \delta - 1$ ,
- (\*3) if  $\max_i |I_i| = 6$  then  $|M(I_1, I_2, \dots, I_s)| \leq 2(\delta - 2)$ .

**Proof.** If  $M(I_1, I_2, \dots, I_s) = \emptyset$  then we are done. Otherwise,  $M(I_a, I_b) \neq \emptyset$  for some distinct  $a, b \in \{1, 2, \dots, s\}$ . By the definition, there is an intermediate path  $L$  between  $I_a$  and  $I_b$ . If  $|L| \geq 2$  then by Lemma 2,

$$|I_a| + |I_b| \geq 2\overline{p} + 2|L| + 4 \geq 10,$$

contradicting (19). Otherwise,  $|L| = 1$  and  $M(I_1, I_2, \dots, I_s) \subseteq E(G)$ . By Lemma 2,  $|I_a| + |I_b| \geq 2\overline{p} + 6 = 8$ . Combining this with (19), we have

$$8 \leq |I_a| + |I_b| \leq 9. \quad (20)$$

Furthermore, if  $|M(I_a, I_b)| \geq 3$  then by Lemma 2,  $|I_a| + |I_b| \geq 2\overline{p} + 8 = 10$ , contradicting (20). So,

$$1 \leq |M(I_i, I_j)| \leq 2 \quad \text{for each distinct } i, j \in \{1, 2, \dots, s\}.$$

Put  $r = |\{i \mid |I_i| \geq 4\}|$ . If  $r \geq 4$  then  $|C| \geq 3(s - 4) + 16 = 3\delta + 1$ , contradicting the hypothesis. Further, if  $r = 0$  then by Lemma 2,  $M(I_1, I_2, \dots, I_s) = \emptyset$ . Let  $1 \leq r \leq 3$ .

**Case a1.**  $r = 3$ .

It follows that  $|I_{a_i}| \geq 4$  ( $i = 1, 2, 3$ ) for some distinct  $a_1, a_2, a_3 \in \{1, 2, \dots, s\}$  and  $|I_i| = 3$  for each  $i \in \{1, 2, \dots, s\} \setminus \{a_1, a_2, a_3\}$ . Since  $s = \delta - 1$  and  $|C| = 3\delta$ , we have  $|I_{a_1}| = |I_{a_2}| = |I_{a_3}| = 4$ , i.e.  $\max |I_i| = 4$ . By Lemma 2,  $|M(I_{a_i}, I_{a_j})| \leq 1$  for each distinct  $i, j \in \{1, 2, 3\}$ . Moreover, we have  $|M(I_i, I_j)| = 0$  if either  $i \notin \{i_1, i_2, i_3\}$  or  $j \notin \{i_1, i_2, i_3\}$ . So,  $|M(I_1, I_2, \dots, I_s)| \leq 3$ .



**Case a2.**  $r = 2$ .

It follows that  $|I_a| \geq 4$  and  $|I_b| \geq 4$  for some  $a, b \in \{1, 2, \dots, s\}$  and  $|I_i| = 3$  for each  $i \in \{1, 2, \dots, s\} \setminus \{a, b\}$ . By (20), we can assume w.l.o.g. that either  $|I_a| = |I_b| = 4$  or  $|I_a| = 5, |I_b| = 4$ .

**Case a2.1.**  $|I_a| = |I_b| = 4$ .

It follows that  $\max_i |I_i| = 4$ . By Lemma 2,  $|M(I_a, I_b)| \leq 1$  and  $M(I_i, I_j) = \emptyset$  if  $\{i, j\} \neq \{a, b\}$ , implying that  $|M(I_1, I_2, \dots, I_s)| \leq 1$ .

**Case a2.2.**  $|I_a| = 5, |I_b| = 4$ .

It follows that  $\max_i |I_i| = 5$ . By Lemma 2, we have  $|M(I_a, I_b)| \leq 2$  and  $|M(I_a, I_i)| \leq 1$  for each  $i \in \{1, 2, \dots, s\} \setminus \{a, b\}$  and  $M(I_i, I_j) = \emptyset$  if  $a \notin \{i, j\}$ . Thus,  $|M(I_1, I_2, \dots, I_s)| \leq \delta - 1$ .

**Case a3.**  $r = 1$ .

It follows that  $|I_a| \geq 4$  for some  $a \in \{1, 2, \dots, s\}$  and  $|I_i| = 3$  for each  $i \in \{1, 2, \dots, s\} \setminus \{a\}$ . By (20),  $4 \leq |I_a| \leq 6$ .

**Case a3.1.**  $|I_a| = 4$ .

It follows that  $\max_i |I_i| = 4$ . By Lemma 2,  $M(I_a, I_i) = \emptyset$  for each  $i \in \{1, 2, \dots, s\} \setminus \{a\}$ , implying that  $|M(I_1, I_2, \dots, I_s)| = 0$ .

**Case a3.2.**  $|I_a| = 5$ .

It follows that  $\max_i |I_i| = 5$ . By Lemma 2,  $|M(I_a, I_i)| \leq 1$  for each  $i \in \{1, 2, \dots, s\} \setminus \{a\}$  and  $M(I_i, I_j) = \emptyset$  if  $a \notin \{i, j\}$ , that is  $|M(I_1, I_2, \dots, I_s)| \leq \delta - 2$ .

**Case a3.3.**  $|I_a| = 6$ .

It follows that  $\max_i |I_i| = 6$ . By Lemma 2,  $|M(I_a, I_i)| \leq 2$  for each  $i \in \{1, 2, \dots, s\} \setminus \{a\}$  and  $M(I_i, I_j) = \emptyset$  if  $a \notin \{i, j\}$ , that is  $|M(I_1, I_2, \dots, I_s)| \leq 2(\delta - 2)$ . Claim 2 is proved.  $\Delta$

Put  $G' = G \setminus M(I_1, I_2, \dots, I_s)$ . As in Case 3.1.2, we can form a graph  $G^*$  by adding in  $G'$  at most  $2|M(I_1, I_2, \dots, I_s)|$  new edges such that  $\delta(G^*) = \delta(G)$ ,  $G^* \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  is disconnected and

$$q(G) \geq q(G^*) - |M(I_1, I_2, \dots, I_s)|. \quad (21)$$

Let  $H_1, H_2, \dots, H_{s+1}$  be the connected components of  $G^* \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  with  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2, \dots, s$ ) and  $V(H_{s+1}) = \{x_1, x_2\}$ . Using notation (5) for  $G^*$ , we have  $h_i \geq 2$  ( $i = 1, 2, \dots, s+1$ ). If  $h_i \geq 6$  for some  $i \in \{1, 2, \dots, s\}$  then  $n \geq 3\delta + 3$ , contradicting (1). Let  $2 \leq h_i \leq 5 < 2\delta - 1$  ( $i = 1, 2, \dots, s+1$ ). It follows that  $(h_i - 2)(2\delta - h_i - 1) \geq 0$  which is equivalent to

$$\frac{h_i(2\delta - h_i + 1)}{2} \geq 2\delta - 1 \quad (i = 1, 2, \dots, s+1). \quad (22)$$

**Case 4.1.1.2.2.1.**  $\max_i |I_i| \leq 4$ .

By (22) and Lemma 3,  $q_i(G^*) \geq 2\delta - 1$  ( $i = 1, 2, \dots, s+1$ ). Hence

$$q(G^*) \geq \sum_{i=1}^{s+1} q_i(G^*) \geq (s+1)(2\delta - 1) = \delta(2\delta - 1).$$

Using (21) and Claim 2, we have

$$q \geq q(G^*) - 3 \geq \delta(2\delta - 1) - 3 \geq \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 4.1.1.2.2.2.**  $\max_i |I_i| = 5$ .

Assume w.l.o.g. that  $\max_i |I_i| = |I_1| = 5$ , i.e.  $4 \leq h_1 \leq 5$ . By (22) and Lemma 3,  $q_i(G^*) \geq 2\delta - 1$  ( $i = 2, \dots, s+1$ ) and

$$q_1(G^*) \geq \frac{h_1(2\delta - h_1 + 1)}{2} \geq 2(2\delta - 3).$$

Hence

$$q(G^*) \geq s(2\delta - 1) + 2(2\delta - 3) = 2\delta^2 + \delta - 5.$$

By (21) and Claim 2,

$$q \geq q(G^*) - (\delta - 1) \geq 2\delta^2 - 4 > \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 4.1.1.2.2.3.**  $\max_i |I_i| = 6$ .

Assume w.l.o.g. that  $\max_i |I_i| = |I_1| = 6$ , that is  $h_1 = 5$ . By (22) and Lemma 3,  $q_i(G^*) \geq 2\delta - 1$  ( $i = 2, \dots, s+1$ ) and

$$q_1(G^*) \geq \frac{h_1(2\delta - h_1 + 1)}{2} = 5(\delta - 2).$$

Hence

$$q(G^*) \geq s(2\delta - 1) + 5(\delta - 2) = 2\delta^2 + 2\delta - 9.$$

By (21) and Claim 2,

$$q \geq q(G^*) - 2(\delta - 2) \geq 2\delta^2 - 5 > \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 4.1.2.**  $\bar{p} = 2$ .

Put  $P = x_1x_3x_2$ . It follows that  $|N_C(x_i)| \geq \delta - 2 > 2$  ( $i = 1, 2$ ). By (1),  $|C| \leq 3\delta + 1 - \bar{p} = 3\delta - 1$ . Combining this with (16), we have

$$3\delta - 3 \leq |C| \leq 3\delta - 1. \quad (23)$$

If  $N_C(x_1) \neq N_C(x_2)$  then by Lemma 1,  $|C| \geq 4\delta - 2\bar{p} = 4\delta - 4$ . By (23),  $4\delta - 4 \leq |C| \leq 3\delta - 1$ , a contradiction. Let  $N_C(x_1) = N_C(x_2)$ . Since  $C$  is extreme, we have  $|I_i| \geq 4$  ( $i = 1, 2, \dots, s$ ). If  $s \geq \delta - 1$  then  $|C| \geq 4s \geq 4\delta - 4 \geq 3\delta$ , contradicting (23). Hence  $s \leq \delta - 2$ . Recalling also that  $s = |N_C(x_1)| \geq \delta - 2$ , we get  $s = \delta - 2$ . It follows that  $x_1x_2 \in E(G)$ . By a symmetric argument,

$$N_C(x_i) = N_C(x_1) \quad (i = 2, 3).$$

**Claim 3.**  $|M(I_1, I_2, \dots, I_s)| \leq 1$ .

**Proof.** If  $M(I_1, I_2, \dots, I_s) = \emptyset$  then we are done. Otherwise,  $|M(I_a, I_b)| \geq 1$  for some  $a, b \in \{1, 2, \dots, s\}$ , i.e. there is an intermediate path  $L$  between  $I_a$  and  $I_b$ . If  $|L| \geq 2$  then by Lemma 2,

$$|I_a| + |I_b| \geq 2\bar{p} + 2|L| + 4 \geq 12.$$

This yields

$$|C| \geq 12 + 4(\delta - 4) = 4\delta - 4 \geq 3\delta + 1,$$

contradicting (23). Otherwise,  $|L| = 1$  and  $M(I_1, I_2, \dots, I_s) \subseteq E(G)$ . By Lemma 2,  $|I_a| + |I_b| \geq 2\bar{p} + 6 = 10$ , implying that  $|C| \geq 10 + 4(\delta - 4) = 4\delta - 6$ . Combining this with (23), we get  $4\delta - 6 \leq |C| \leq 3\delta - 1$ , i.e.  $\delta \leq 5$ . Since  $\delta \geq 5$ , we have

$$\delta = 5, \quad s = 3, \quad |C| = 3\delta - 1 = 14, \quad |I_a| + |I_b| = 10,$$

$$|I_i| = 4 \text{ for each } i \in \{1, 2, \dots, s\} \setminus \{a, b\}.$$

Assume w.l.o.g. that  $a = 1$  and  $b = 2$ . By Lemma 2,  $|M(I_1, I_2)| = 1$  and  $M(I_1, I_3) = M(I_2, I_3) = \emptyset$ , i.e.  $|M(I_1, I_2, \dots, I_s)| = 1$ . Claim 3 is proved.  $\Delta$

Put  $G' = G \setminus M(I_1, I_2, \dots, I_s)$ . As in Case 3.1.2, form a graph  $G^*$  by adding in  $G'$  at most  $2|M(I_1, I_2, \dots, I_s)|$  appropriate new edges such that  $\delta(G^*) = \delta(G)$ ,  $G^* \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  is disconnected and

$$q(G) \geq q(G^*) - |M(I_1, I_2, \dots, I_s)|.$$

Let  $H_1, H_2, \dots, H_{s+1}$  be the connected components of  $G^* \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  with  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2, \dots, s$ ) and  $V(H_{s+1}) = \{x_1, x_2, x_3\}$ . Using notation (5) for  $G^*$ , we have  $h_i \geq 3$  ( $i = 1, 2, \dots, s + 1$ ). If  $h_i \geq 6$  for some  $i \in \{1, 2, \dots, s\}$  then

$$|C| \geq 6 + 3(s - 1) + |\{\xi_1, \xi_2, \dots, \xi_s\}| = 4\delta - 5 \geq 3\delta,$$

contradicting (23). Let  $3 \leq h_i \leq 5$  ( $i = 1, 2, \dots, s + 1$ ). By Lemma 3,

$$q_i(G^*) \geq \frac{h_i(2\delta - h_i + 1)}{2} \geq 3(\delta - 1) \quad (i = 1, 2, \dots, s + 1),$$

implying that

$$q(G^*) \geq \sum_{i=1}^{s+1} q_i(G^*) \geq 3(s + 1)(\delta - 1) = 3(\delta - 1)^2.$$

By Claim 3,

$$q \geq q(G^*) - 1 \geq 3(\delta - 1)^2 - 1 > \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 4.1.3.**  $\bar{p} = 3$ .

Put  $P = x_1x_4x_3x_2$ . It follows that  $|N_C(x_i)| \geq \delta - 3 \geq 2$  ( $i = 1, 2$ ). By (1),  $|C| \leq 3\delta + 1 - \bar{p} = 3\delta - 2$ . Combining this with (16), we have

$$3\delta - 3 \leq |C| \leq 3\delta - 2. \quad (24)$$

If  $N_C(x_1) \neq N_C(x_2)$  then by Lemma 1,

$$|C| \geq 4\delta - 2\bar{p} = 4\delta - 6 \geq 3\delta - 1,$$

contradicting (24). Let  $N_C(x_1) = N_C(x_2)$ . Clearly,  $|I_i| \geq 5$  ( $i = 1, 2, \dots, s$ ). If  $s \geq \delta - 2$  then  $|C| \geq 5s \geq 5\delta - 10 > 3\delta - 1$ , contradicting (24). Hence  $s \leq \delta - 3$ . Observing also that  $s = |N_C(x_1)| \geq \delta - 3$ , we get  $s = \delta - 3$ . It follows that  $x_1x_2 \in E(G)$ . By symmetric arguments,  $N_C(x_i) = N_C(x_1)$  ( $i = 2, 3, 4$ ).

**Claim 4.**  $M(I_1, I_2, \dots, I_s) = \emptyset$ .

**Proof.** Assume to the contrary, i.e.  $M(I_1, I_2, \dots, I_s) \neq \emptyset$ . It means that  $M(I_a, I_b) \neq \emptyset$  for some distinct  $a, b \in \{1, 2, \dots, s\}$ . By Lemma 2,

$$|I_a| + |I_b| \geq 4\delta - 2\bar{p} = 4\delta - 6,$$

implying that  $|C| \geq (4\delta - 6) + 5(s - 2) = 9\delta - 31$ . Combining this with (24), we get  $9\delta - 31 \leq |C| \leq 3\delta - 2$ , a contradiction. Claim 4 is proved.  $\Delta$

By Claim 4,  $G \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  is disconnected. Let  $H_1, H_2, \dots, H_{s+1}$  be the connected components of  $G \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  with  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2, \dots, s$ ) and  $V(H_{s+1}) = V(P)$ . By notation (5), we have  $h_i \geq 4$  ( $i = 1, 2, \dots, s + 1$ ). If  $h_i \geq 8$  for some  $i \in \{1, 2, \dots, s\}$  then

$$|C| \geq 8 + 4(\delta - 4) + s = 5\delta - 11 \geq 3\delta - 1,$$

contradicting (24). Let  $4 \leq h_i \leq 7$  ( $i = 1, 2, \dots, s + 1$ ). By Lemma 3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} \geq 2(2\delta - 3) \quad (i = 1, 2, \dots, s + 1).$$

Hence

$$\begin{aligned} q &\geq \sum_{i=1}^{s+1} q_i \geq 2(s+1)(2\delta - 3) \\ &= 2(\delta - 2)(2\delta - 3) \geq \frac{3(\delta - 1)(\delta + 2)}{2}. \end{aligned}$$

**Case 4.1.4.**  $\bar{p} = 4$ .

Put  $P = x_1x_5x_4x_3x_2$ . By (1),  $|C| \leq 3\delta + 1 - \bar{p} = 3\delta - 3$ , and by (16),  $|C| \geq 3\delta - 3$ . It follows that

$$|C| = 3\delta - 3, \quad n = 3\delta + 2, \quad V(G) = V(C \cup P). \quad (25)$$

If  $\delta \leq 6$  then

$$q \geq \frac{n\delta}{2} = \frac{(3\delta + 2)\delta}{2} \geq \frac{3(\delta - 1)(\delta + 2)}{2}.$$

Let  $\delta \geq 7$ , implying that  $|N_C(x_i)| > 2$  ( $i = 1, 2$ ). If  $N_C(x_1) \neq N_C(x_2)$  then by Lemma 1,

$$|C| \geq 4\delta - 2\bar{p} = 4\delta - 8 \geq 3\delta - 2,$$

contradicting (25). Let  $N_C(x_1) = N_C(x_2)$ . Clearly,  $|I_i| \geq 6$  ( $i = 1, 2, \dots, s$ ). If  $s \geq \delta - 3$  then

$$|C| \geq (\bar{p} + 2)s \geq 6(\delta - 3) \geq 3\delta - 2,$$

contradicting (25). Let  $s \leq \delta - 4$ . On the other hand,  $s \geq |N(x_1)| - \bar{p} \geq \delta - 4$ , implying that  $s = \delta - 4$ . It follows that  $x_1x_2 \in E(G)$ . By symmetric arguments,  $N_C(x_i) = N_C(x_1)$  ( $i = 2, 3, 4, 5$ ). If  $M(I_a, I_b) \neq \emptyset$  for some distinct elementary segments  $I_a, I_b$ , then by Lemma 2,

$$|I_a| + |I_b| \geq 4\delta - 2\bar{p} = 4\delta - 8.$$

Hence

$$|C| \geq 4\delta - 8 + 6(s - 2) = 10\delta - 44 \geq 3\delta - 2,$$

contradicting (25). Otherwise,  $M(I_1, I_2, \dots, I_s) = \emptyset$ , i.e.  $G \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  is disconnected. Let  $H_1, H_2, \dots, H_{s+1}$  be the connected components of  $G \setminus \{\xi_1, \xi_2, \dots, \xi_s\}$  with  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2, \dots, s$ ) and  $V(H_{s+1}) = V(P)$ . By notation (5),  $h_i \geq 5$  ( $i = 1, 2, \dots, s + 1$ ). If  $h_i \geq 6$  for some  $i \in \{1, 2, \dots, s\}$  then

$$|C| \geq 6 + 5(s - 1) + s = 6\delta - 23 \geq 3\delta - 2,$$

contradicting (25). So,  $h_i = 5$  ( $i = 1, 2, \dots, s + 1$ ). By Lemma 3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = 5(\delta - 2) \quad (i = 1, 2, \dots, s + 1),$$

implying that

$$q \geq \sum_{i=1}^{s+1} q_i \geq 5(s + 1)(\delta - 2) = 5(\delta - 3)(\delta - 2) > \frac{3(\delta - 1)(\delta - 2)}{2}.$$

**Case 4.2.**  $\delta - 2 \leq \bar{p} \leq \delta + 1$ .

**Case 4.2.1.**  $\bar{p} = \delta - 2$ .

It follows that  $|N_C(x_i)| \geq 2$  ( $i = 1, 2$ ). By (1),

$$|C| \leq 3\delta + 1 - \bar{p} = 2\delta + 3. \tag{26}$$

If  $N_C(x_1) \neq N_C(x_2)$  then by Lemma 1,  $|C| \geq 4\delta - 2\bar{p} = 2\delta + 4$ , contradicting (26). Let  $N_C(x_1) = N_C(x_2)$ . If  $s \geq 3$  then  $|C| \geq s(\bar{p} + 2) \geq 3\delta \geq 2\delta + 4$ , again contradicting (26). Let  $s = 2$ . It follows that  $x_1x_2 \in E(G)$ . By symmetric arguments,  $N_C(y) = N_C(x_1) = \{\xi_1, \xi_2\}$  for each  $y \in V(P)$ . Clearly,  $|I_i| \geq \bar{p} + 2 = \delta$  ( $i = 1, 2$ ).

**Case 4.2.1.1.**  $M(I_1, I_2) = \emptyset$ .

It follows that  $G \setminus \{\xi_1, \xi_2\}$  is disconnected. Let  $H_1, H_2, \dots, H_t$  be the connected components of  $G \setminus \{\xi_1, \xi_2\}$  with  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2$ ) and  $V(P) \subset V(H_3)$ . Since  $G[V(P)]$  is hamiltonian, we have  $V(H_3) = V(P)$ . By notation (5),  $h_i \geq |I_i| - 1 \geq \delta - 1$  ( $i = 1, 2$ ) and  $h_3 = \delta - 1$ . If  $h_i \geq \delta + 3$  for some  $i \in \{1, 2\}$  then

$$|C| \geq (\delta + 3) + (\delta - 1) + |\{\xi_1, \xi_2\}| = 2\delta + 4,$$

contradicting (26). So,  $\delta - 1 \leq h_i \leq \delta + 2$  ( $i = 1, 2, 3$ ). By Lemma 3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} \geq \frac{(\delta - 1)(\delta + 2)}{2} \quad (i = 1, 2, 3).$$

Hence,

$$q \geq \sum_{i=1}^3 q_i \geq \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 4.2.1.2.**  $M(I_1, I_2) \neq \emptyset$ .

By the definition, there is an intermediate path  $L$  between  $I_1$  and  $I_2$ . If  $|L| \geq 2$  then by Lemma 2,

$$|C| = |I_1| + |I_2| \geq 2\overline{p} + 2|L| + 4 \geq 2\delta + 4,$$

contradicting (26). Otherwise,  $M(I_1, I_2) \subseteq E(G)$ . Further, if  $|M(I_1, I_2)| \geq 3$  then by Lemma 2,

$$|C| = |I_1| + |I_2| \geq 2\overline{p} + 8 = 2\delta + 4,$$

contradicting (26). Thus  $|M(I_1, I_2)| \leq 2$ .

**Case 4.2.1.2.1.**  $|M(I_1, I_2)| = 1$ .

Put  $G' = G \setminus M(I_1, I_2)$ . As in Case 3.1.2, form a graph  $G^*$  by adding at most two new edges in  $G'$  such that  $\delta(G^*) = \delta(G)$ ,  $G^* \setminus \{\xi_1, \xi_2\}$  is disconnected and  $q(G) \geq q(G^*) - 1$ . Let  $H_1, H_2, \dots, H_t$  be the connected components of  $G^* \setminus \{\xi_1, \xi_2\}$  with  $V(I_i^*) \subseteq V(H_i)$  ( $i = 1, 2$ ) and  $V(P) = V(H_3)$ . Using notation (5) for  $G^*$ , as in Case 4.2.1.1, we have  $\delta - 1 \leq h_i \leq \delta + 2$  ( $i = 1, 2, 3$ ). By Lemma 2,  $|I_1| + |I_2| \geq 2\overline{p} + 6 = 2\delta + 2$ . It means that  $\max_i |I_i| \geq \delta + 1$ , i.e.  $\max_i h_i \geq \delta$ . Assume w.l.o.g. that  $h_1 \geq \delta$ . By Lemma 3,

$$q_1(G^*) \geq \frac{h_1(2\delta - h_1 + 1)}{2} \geq \frac{\delta(\delta + 1)}{2},$$

$$q_i(G^*) \geq \frac{h_i(2\delta - h_i + 1)}{2} \geq \frac{(\delta - 1)(\delta + 2)}{2} \quad (i = 2, 3),$$

implying that

$$q(G^*) \geq \frac{\delta(\delta + 1)}{2} + (\delta - 1)(\delta + 2).$$

Hence,

$$q \geq q(G^*) - 1 \geq \frac{\delta(\delta+1)}{2} + (\delta-1)(\delta+2) - 1 \geq \frac{3(\delta-1)(\delta+2)}{2}.$$

**Case 4.2.1.2.2.**  $|M(I_1, I_2)| = 2$ .

By Lemma 2,

$$|C| = |I_1| + |I_2| \geq 2\bar{p} + 7 = 2\delta + 3.$$

By (26),  $|C| = 2\delta + 3$  and  $V(G) = V(P \cup C)$ . Put  $G' = G \setminus M(I_1, I_2)$ . As in Case 3.1.2, form a graph  $G^*$  by adding at most four new edges in  $G'$  such that  $\delta(G^*) = \delta(G)$ ,  $G^* \setminus \{\xi_1, \xi_2\}$  is disconnected and  $q(G) \geq q(G^*) - 2$ . Let  $H_1, H_2, H_3$  be the connected components of  $G^* \setminus \{\xi_1, \xi_2\}$  with  $V(H_i) = V(I_i^*)$  ( $i = 1, 2$ ) and  $V(H_3) = V(P)$ . Using notation (5) for  $G^*$ , we have as in Case 4.2.1.1,  $\delta-1 \leq h_i \leq \delta+2$  ( $i = 1, 2, 3$ ). Since  $|I_i| \geq \delta$  and  $|C| = |I_1| + |I_2| = 2\delta+3$ , we can assume w.l.o.g. that either  $|I_1| = \delta+2, |I_2| = \delta+1$  or  $|I_1| = \delta+3, |I_2| = \delta$ .

**Case 4.2.1.2.2.1.**  $|I_1| = \delta+2, |I_2| = \delta+1$ .

It follows that  $h_1 = \delta+1, h_2 = \delta$  and  $h_3 = \delta-1$ . By Lemma 3,

$$\begin{aligned} q_i(G^*) &\geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{\delta(\delta+1)}{2} \quad (i = 1, 2), \\ q_3(G^*) &\geq \frac{h_3(2\delta - h_3 + 1)}{2} = \frac{(\delta-1)(\delta+2)}{2}. \end{aligned}$$

Hence

$$q \geq \sum_{i=1}^3 q_i(G^*) - 2 = \frac{3(\delta-1)(\delta+2)}{2}.$$

**Case 4.2.1.2.2.2.**  $|I_1| = \delta+3, |I_2| = \delta$ .

Let  $M(I_1, I_2) = \{e_1, e_2\}$ , where

$$e_1 = y_1 z_1, e_2 = y_2 z_2, \{y_1, y_2\} \subseteq V(I_1^*), \{z_1, z_2\} \subseteq V(I_2^*).$$

If  $y_1 \neq y_2$  and  $z_1 \neq z_2$  then by Lemma 2,

$$|I_1| + |I_2| \geq 2\bar{p} + 8 = 2(\delta-2) + 8 = 2\delta + 4,$$

contradicting (26). Let either  $y_1 \neq y_2$  and  $z_1 = z_2$  or  $y_1 = y_2$  and  $z_1 \neq z_2$ .

**Case 4.2.1.2.2.2.1.**  $y_1 \neq y_2$  and  $z_1 = z_2$ .

Assume w.l.o.g. that  $y_1, y_2$  occur on  $I_1$  in this order. If  $y_2 = y_1^+$  then

$$|C| \geq |\xi_1 \overrightarrow{C} y_1 z_1 y_2 \overrightarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1| = 2\delta + 4,$$

contradicting (26). Let  $y_2 \neq y_1^+$ , i.e.  $|y_1 \overrightarrow{C} y_2| \geq 2$ . Put

$$C' = \xi_1 \overrightarrow{C} y_2 z_1 \overleftarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1,$$

$$C'' = \xi_1 \overleftarrow{C} z_1 y_1 \overrightarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1.$$

Clearly,

$$\begin{aligned}|C| &\geq |C'| = |\xi_1 \vec{C} y_1| + |y_1 \vec{C} y_2| + 1 + |\xi_2 \vec{C} z_1| + \bar{p} + 2, \\ |C| &\geq |C''| = |\xi_1 \overleftarrow{C} z_1| + |y_1 \vec{C} y_2| + |y_2 \vec{C} \xi_2| + 1 + \bar{p} + 2.\end{aligned}$$

By summing, we get

$$\begin{aligned}2|C| &\geq (|\xi_1 \vec{C} y_1| + |y_1 \vec{C} y_2| + |y_2 \vec{C} \xi_2| + |\xi_2 \vec{C} z_1| + |z_1 \vec{C} \xi_1|) + |y_1 \vec{C} y_2| + 2 + 2\delta \\ &= |C| + |y_1 \vec{C} y_2| + 2\delta + 2 \geq |C| + 2\delta + 4.\end{aligned}$$

Hence  $|C| \geq 2\delta + 4$ , contradicting (26).

**Case 4.2.1.2.2.2.2.**  $y_1 = y_2$  and  $z_1 \neq z_2$ .

Assume w.l.o.g. that  $z_2, z_1$  occur on  $I_2$  in this order.

**Case 4.2.1.2.2.2.2.1.**  $\delta \geq 6$ .

If  $|\xi_1 \vec{C} y_1| \geq \delta - 1$  and  $|y_1 \vec{C} \xi_2| \geq \delta - 1$  then  $|I_1| \geq 2\delta - 2 \geq \delta + 4$ , contradicting the hypothesis. Thus, we can assume w.l.o.g. that  $|\xi_1 \vec{C} y_1| \leq \delta - 2$ . If  $y_1^- = \xi_1$  then

$$|\xi_1 \overleftarrow{C} z_2 y_1 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1| \geq 2\delta + 5,$$

contradicting (26). Let  $y_1^- \neq \xi_1$ , that is  $y_1^- \in V(I_1^*)$ . Since  $|M(I_1, I_2)| = 2$ , we have  $N(y_1^-) \subset V(I_1)$ . If  $N(y_1^-) \cap V(y_1^+ \vec{C} \xi_2^-) = \emptyset$  then  $|N(y_1^-)| \leq \delta - 1$ , a contradiction. Otherwise,  $y_1^- w \in E(G)$  for some  $w \in V(y_1^+ \vec{C} \xi_2^-)$ . Put

$$\begin{aligned}R &= \xi_1 \vec{C} y_1^- w \overleftarrow{C} y_1 \\ C' &= \xi_1 \vec{R} y_1 z_1 \overleftarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1, \\ C'' &= \xi_1 \overleftarrow{C} z_2 y_1 \vec{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1.\end{aligned}$$

Clearly,  $|R| \geq |\xi_1 \vec{C} y_1| + 1$  and

$$\begin{aligned}|C| &\geq |C'| = |R| + 1 + |z_1 \overleftarrow{C} \xi_2| + (\bar{p} + 2) \geq |\xi_1 \vec{C} y_1| + |z_1 \overleftarrow{C} \xi_2| + \delta + 2, \\ |C| &\geq |C''| = |\xi_1 \overleftarrow{C} z_1| + 2 + |y_1 \vec{C} \xi_2| + (\bar{p} + 2).\end{aligned}$$

By summing, we get

$$2|C| \geq (|\xi_1 \vec{C} y_1| + |y_1 \vec{C} \xi_2| + |\xi_2 \vec{C} z_1| + |z_1 \vec{C} \xi_1|) + 2\delta + 4 = |C| + 2\delta + 4.$$

Hence  $|C| \geq 2\delta + 4$ , contradicting (26).

**Case 4.2.1.2.2.2.2.2.**  $\delta = 5$ .

It follows that

$$|I_1| = \delta + 3 = 8, \quad |I_2| = \delta = 5, \quad |C| = 2\delta + 3 = 13.$$



If either  $|\xi_1 \vec{C} y_1| \leq \delta - 2 = 3$  or  $|y_1 \vec{C} \xi_2| \leq \delta - 2 = 3$  then we can argue as in Case 4.2.1.2.2.2.2.1. Otherwise,  $|\xi_1 \vec{C} y_1| = |y_1 \vec{C} \xi_2| = 4$ . If  $|z_1 \overleftarrow{C} \xi_2| \geq 4$  then

$$|\xi_1 \vec{C} y_1 z_1 \overleftarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1| \geq 14 > |C|,$$

a contradiction. Let  $|z_1 \overleftarrow{C} \xi_2| \leq 3$ . Analogously,  $|\xi_1 \overleftarrow{C} z_2| \leq 3$ , implying that  $I_2 = \xi_1 z_1^+ z_1 z_2 \xi_2^+ \xi_2$ . If  $z_1^+ z_2 \in E(G)$  then

$$|\xi_1 \vec{C} y_1 z_1 z_1^+ z_2 \overleftarrow{C} \xi_2 x_2 \overleftarrow{P} x_1 \xi_1| = 14 > |C|,$$

a contradiction. So,  $N(z_1^+) \subseteq \{\xi_1, \xi_2, z_1, \xi_2^+\}$ , again a contradiction, since  $|N(z_1^+)| \geq \delta = 5$ .

**Case 4.2.2.**  $\overline{p} = \delta - 1$ .

By (1),

$$|C| \leq 3\delta + 1 - \overline{p} = 2\delta + 2. \quad (27)$$

It follows that  $|N_C(x_i)| \geq 1$  ( $i = 1, 2$ ).

**Case 4.2.2.1.**  $|N_C(x_i)| \geq 2$  ( $i = 1, 2$ ).

If  $N_C(x_1) \neq N_C(x_2)$  then by Lemma 1,  $|C| \geq 2\overline{p} + 8 = 2\delta + 6$ , contradicting (27). Let  $N_C(x_1) = N_C(x_2)$ . If  $s \geq 3$  then

$$|C| \geq s(\overline{p} + 2) \geq 3(\delta + 1) > 2\delta + 2,$$

contradicting (27). Let  $s = 2$ . It follows that

$$|C| = 2\delta + 2, |I_1| = |I_2| = \delta + 1, V(G) = V(C \cup P).$$

Assume that  $yz \in E(G)$  for some  $y \in V(P)$  and  $z \in V(C) \setminus \{\xi_1, \xi_2\}$ . Assume w.l.o.g. that  $z \in V(I_1^*)$ . Since  $\overline{p} = \delta - 1 \geq 4$ , we can assume w.l.o.g. that  $|x_1 \vec{P} y| \geq 2$ . If  $x_2 w \in E(G)$  for some  $w \in \{y^-, y^{-2}\}$  then

$$|\xi_1 \vec{C} z| \geq |\xi_1 x_1 \vec{P} w x_2 \overleftarrow{P} y z| \geq \delta.$$

Observing also that  $|z \vec{C} \xi_2| \geq 2$ , we have  $|I_1| \geq \delta + 2$ , a contradiction. Otherwise,  $d(x_2) \leq \delta - 1$ , a contradiction. So,  $N_C(y) \subseteq \{\xi_1, \xi_2\}$  for each  $y \in V(P)$ . On the other hand, by Lemma 2,  $M(I_1, I_2) = \emptyset$  and hence  $G \setminus \{\xi_1, \xi_2\}$  is disconnected. Let  $H_1, H_2, H_3$  be the connected components of  $G \setminus \{\xi_1, \xi_2\}$  with  $V(H_i) = V(I_i^*)$  ( $i = 1, 2$ ) and  $V(H_3) = V(P)$ . By notation (5),  $h_i = \delta$  ( $i = 1, 2, 3$ ). By Lemma 3,

$$q_i \geq \frac{h_i(2\delta - h_i + 1)}{2} = \frac{\delta(\delta + 1)}{2} \quad (i = 1, 2, 3),$$

implying that

$$q \geq \sum_{i=1}^3 q_i \geq \frac{3(\delta^2 + \delta)}{2} > \frac{3(\delta - 1)(\delta + 2)}{2}.$$

**Case 4.2.2.2.** Either  $|N_C(x_1)| = 1$  or  $|N_C(x_2)| = 1$ .

Assume w.l.o.g. that  $|N_C(x_1)| = 1$ . It follows that  $V(P) \setminus \{x_1\} \subset N(x_1)$ . Put  $N_C(x_1) = \{y_1\}$ .

**Case 4.2.2.2.1.**  $N_C(x_2) \neq N_C(x_1)$ .

It follows that  $x_2 y_2 \in E(G)$  for some  $y_2 \in V(C) \setminus \{y_1\}$ . Clearly,  $|y_1 \vec{C} y_2| \geq \delta + 1$  and  $|y_2 \vec{C} y_1| \geq \delta + 1$ . Hence

$$|y_1 \vec{C} y_2| = |y_2 \vec{C} y_1| = \delta + 1, \quad |C| = 2\delta + 2, \quad V(G) = V(C \cup P). \quad (28)$$

If  $s \geq 3$  then there are at least two elementary segments on  $C$  of length at least  $\delta + 1$ . It means that  $|C| > 2\delta + 2$ , contradicting (28). Let  $s = 2$ , i.e.  $N_C(x_1) \cup N_C(x_2) = \{y_1, y_2\} = \{\xi_1, \xi_2\}$ . Assume that  $zw \in E(G)$  for some  $z \in V(P)$  and  $w \in V(C) \setminus \{y_1, y_2\}$ , and assume w.l.o.g. that  $w \in y_1 \vec{C} y_2$ . Since  $V(P) \setminus \{x_1\} \subset N(x_1)$ , we have  $x_1 z^+ \in E(G)$ . Further, since  $C$  is extreme,

$$|w \vec{C} y_2| \geq |wz \overleftarrow{P} x_1 z^+ \vec{P} x_2 y_2| \geq \delta + 1.$$

Hence,  $|C| > 2\delta + 2$ , contradicting (28). Thus,  $N(z) \subseteq \{y_1, y\}$  for each  $z \in V(P)$ . On the other hand, by Lemma 2,  $M(I_1, I_2) = \emptyset$ . Further, we can argue as in Case 4.2.2.1.

**Case 4.2.2.2.2.**  $N_C(x_2) = N_C(x_1) = \{y_1\}$ .

It follows that

$$N(x_i) = (V(P) \setminus \{x_i\}) \cup \{y_1\} \quad (i = 1, 2).$$

Since  $\kappa \geq 2$ , there is a path  $R = z \vec{R} w$  such that  $z \in V(P)$  and  $w \in V(C) \setminus \{y_1\}$ . Since  $N_C(x_1) = N_C(x_2) = \{y_1\}$ , we have  $z \notin \{x_1, x_2\}$ . Then

$$|y_1 x_1 \vec{P} z^- x_2 \overleftarrow{P} zw| = \delta + 1,$$

and we can argue as in Case 4.2.2.1.

**Case 4.2.3.**  $\bar{p} = \delta$ .

By (1),  $|C| \leq 3\delta + 1 - \bar{p} = 2\delta + 1$ . If  $|Q| \geq \delta + 1$  then by (2),  $|C| \geq 2|Q| \geq 2\delta + 2$ , a contradiction. Let

$$|Q| \leq \delta. \quad (29)$$

**Case 4.2.3.1.**  $x_1 x_2 \notin E(G)$

It follows that  $|N_C(x_i)| \geq 1$  ( $i = 1, 2$ ). If  $|N_C(x_i)| \geq 2$  for some  $i \in \{1, 2\}$  then clearly  $|Q| \geq \bar{p} + 2 = \delta + 2$ , contradicting (29). Let  $|N_C(x_1)| = |N_C(x_2)| = 1$ . Further, if  $N_C(x_1) \neq N_C(x_2)$  then again  $|Q| \geq \delta + 2$ , contradicting (29). Let  $N_C(x_1) = N_C(x_2) = \{z_1\}$  for some  $z_1 \in V(C)$ . Since  $\kappa \geq 2$ , there is a path  $L = y z_2$  connecting  $P$  and  $C$  such that  $y \in V(P)$  and  $z_2 \in V(C) \setminus \{z_1\}$ . Clearly,  $y \notin \{x_1, x_2\}$ . If  $x_2 y^- \in E(G)$  then

$$|Q| \geq |z_1 x_1 \vec{P} y^- x_2 \overleftarrow{P} y z_2| = \delta + 2,$$

contradicting (29). Let  $x_2y^- \notin E(G)$ . Further, if  $y^- \neq x_1$  then recalling that  $x_2x_1 \notin E(G)$ , we conclude that  $|N_C(x_2)| \geq 2$ , a contradiction. Otherwise,  $y^- = x_1$  and  $|Q| \geq |z_1x_2\overrightarrow{P}y z_2| = \delta + 1$ , contradicting (29).

**Case 4.2.3.2.**  $x_1x_2 \in E(G)$ .

Put  $C' = x_1\overrightarrow{P}x_2x_1$ . Since  $\kappa \geq 2$ , there are two disjoint paths  $L_1, L_2$  connecting  $C'$  and  $C$ . Further, since  $P$  is extreme,  $|L_1| = |L_2| = 1$ . Let  $L_1 = y_1z_1$  and  $L_2 = y_2z_2$ , where,  $y_1, y_2 \in V(C')$  and  $z_1, z_2 \in V(C)$ . Since  $C'$  is a Hamilton cycle in  $G[V(P)]$  and  $|C'| \geq \delta + 1 \geq 6$ , we can assume that  $P$  is chosen such that  $x_1 = y_1$  and  $|x_1\overrightarrow{P}y_2| \geq 3$ . If  $x_2v \in E(G)$  for some  $v \in \{y_2^{-1}, y_2^{-2}\}$  then

$$|Q| \geq |z_1x_1\overrightarrow{P}vx_2\overleftarrow{P}y_2z_2| \geq \delta + 1,$$

contradicting (29). Otherwise,  $|N_C(x_2)| \geq 2$ , implying that  $x_2z_3 \in E(G)$  for some  $z_3 \in V(C) \setminus \{z_1\}$ . Then

$$|Q| \geq |z_1x_1\overrightarrow{P}x_2z_3| \geq \delta + 2,$$

again contradicting (29).

**Case 4.2.4.**  $\overline{p} = \delta + 1$ .

By (1),  $|C| \leq 3\delta + 1 - \overline{p} = 2\delta$ . Recalling (14), we get  $|C| = 2\delta$  and  $V(G) = V(C \cup P)$ . If  $|Q| \geq \delta + 1$  then by (2),  $|C| \geq 2|Q| \geq 2\delta + 2$ , a contradiction. Let

$$|Q| \leq \delta. \tag{30}$$

**Case 4.2.4.1.**  $x_1x_2 \in E(G)$ .

Put  $C' = x_1\overrightarrow{P}x_2x_1$ . Since  $\kappa \geq 2$ , there are two disjoint edges  $z_1w_1$  and  $z_2w_2$  connecting  $C'$  and  $C$  such that  $z_1, z_2 \in V(C')$  and  $w_1, w_2 \in V(C)$ . Since  $C'$  is a Hamilton cycle in  $G[V(P)]$  and  $|C'| \geq \delta + 2 \geq 7$ , we can assume w.l.o.g. that  $P$  is chosen such that  $z_1 = x_1$  and  $|x_1\overrightarrow{P}z_2| \geq 4$ . If  $x_2v \in E(G)$  for some  $v \in \{z_2^{-1}, z_2^{-2}, z_2^{-3}\}$  then

$$|Q| \geq |w_1x_1\overrightarrow{P}vx_2\overleftarrow{P}z_2w_2| \geq \delta + 1,$$

contradicting (30). Now let  $x_2v \notin E(G)$  for each  $v \in \{z_2^{-1}, z_2^{-2}, z_2^{-3}\}$ . It follows that  $|N_C(x_2)| \geq 2$ , i.e.  $x_2w_3 \in E(G)$  for some  $w_3 \in V(C) \setminus \{w_1\}$ . But then  $|Q| \geq |w_1x_1\overrightarrow{P}x_2w_3| = \delta + 3$ , contradicting (30).

**Case 4.2.4.2.**  $x_1x_2 \notin E(G)$ .

If  $d_P(x_1) + d_P(x_2) \geq |V(P)| = \overline{p} + 2$  then by Theorem F,  $G[V(P)]$  is hamiltonian and we can argue as in Case 4.2.4.1. Otherwise,

$$d_C(x_1) + d_C(x_2) \geq \delta - 1 \geq 4. \tag{31}$$

Assume w.l.o.g. that  $d_C(x_1) \geq d_C(x_2)$ .

**Case 4.2.4.2.1.**  $d_C(x_2) = 0$ .

It follows that  $N(x_2) = V(P) \setminus \{x_2\}$ . By (31),  $d_C(x_1) \geq 4$ . Put  $C' = x_1^+ \vec{P} x_2 x_1^+$ . Since  $\kappa \geq 2$ , there is a path  $L = z \vec{L} w$  connecting  $C'$  and  $C$  such that  $z \in V(C') \setminus \{x_1^+\}$  and  $w \in V(C)$ . If  $x_1 \in V(L)$ , i.e.  $x_1 z \in E(G)$ , then  $x_1 \vec{P} z^- x_2 \overleftarrow{P} z x_1$  is a Hamilton cycle in  $G[V(P)]$  and we can argue as in Case 4.2.4.1. Let  $x_1 \notin V(L)$ . Since  $V(G) = V(C \cup P)$ , we have  $L = zw$ . Further, since  $d_C(x_1) \geq 4$ , we have  $x_1 w_1 \in E(G)$  for some  $w_1 \in V(C) \setminus \{w\}$ . Hence,

$$|Q| \geq |w_1 x_1 \vec{P} z^- x_2 \overleftarrow{P} zw| = \delta + 3,$$

contradicting (30).

**Case 4.2.4.2.2.**  $d_C(x_2) = 1$ .

Let  $N_C(x_2) = \{w_1\}$ . By (31),  $d_C(x_1) \geq 3$ , i.e.  $x_1 w \in E(G)$  for some  $w \in V(C) \setminus \{w_1\}$ . Hence

$$|Q| \geq |w x_1 \vec{P} x_2 w_1| = \delta + 3,$$

contradicting (30).

**Case 4.2.4.2.3.**  $d_C(x_2) \geq 2$ .

Since  $d_C(x_1) \geq d_C(x_2)$ , we have  $d_C(x_1) \geq 2$ . Hence  $|Q| \geq \bar{p} + 2 = \delta + 3$ , contradicting (30). ■

**Proof of Theorem 1.** Let  $G$  be a 2-connected graph,  $C$  a longest cycle in  $G$  and  $P = x_1 \vec{P} x_2$  a longest path in  $G \setminus C$  of length  $\bar{p}$ . If  $\bar{p} = 0$  then  $C$  is a dominating cycle and we are done. Let  $\bar{p} \geq 1$ .

**Case 1.**  $\delta = 2$  and  $q \leq 8$ .

Since  $\kappa \geq 2$  and  $\bar{p} \geq 1$ , there exist a path  $Q = \xi \vec{Q} \eta$  such that  $|Q| \geq 3$  and  $V(Q) \cap V(C) = \{\xi, \eta\}$ . Further, since  $C$  is extreme, we have  $|C| = |y \vec{C} z| + |z \vec{C} y| \geq 2|Q| \geq 6$  and therefore,  $q \geq |C| + |Q| \geq 9$ , contradicting the hypothesis.

**Case 2.**  $\delta \geq 3$  and  $q \leq (3(\delta - 1)(\delta + 2) - 1)/2$ .

Since

$$q = \frac{1}{2} \sum_{u \in V(G)} d(u) \geq \frac{\delta n}{2},$$

we have  $\delta n/2 \leq (3(\delta - 1)(\delta + 2) - 1)/2$ , which is equivalent to

$$\delta \geq \frac{n-2}{3} - \frac{1}{3} + \frac{7}{3\delta}.$$

If  $n = 3t$  for some integer  $t$ , then

$$\delta \geq \frac{3t-2}{3} - \frac{1}{3} + \frac{7}{3\delta} = t - 1 + \frac{7}{3\delta},$$

implying that  $\delta \geq t = n/3 > (n-2)/3$ . If  $n = 3t + 1$  for some integer  $t$ , then

$$\delta \geq \frac{3t-1}{3} - \frac{1}{3} + \frac{7}{3\delta} = t - \frac{2}{3} + \frac{7}{3\delta},$$

implying that  $\delta \geq t = (n-1)/3 > (n-2)/3$ . Finally, if  $n = 3t + 2$  for some integer  $t$ , then

$$\delta \geq \frac{3t}{3} - \frac{1}{3} + \frac{7}{3\delta} = t - \frac{1}{3} + \frac{7}{3\delta},$$

implying that  $\delta \geq t = (n-2)/3$ . So,  $\delta \geq (n-2)/3$ , in any case. By Lemma 4, each longest cycle in  $G$  is a dominating cycle. ■

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